

## Theoretical Investigation of a Cascaded Fin Thermal Behavior Using MATLAB and Symbolic Math Toolbox

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### ABSTRACT

Thermal behavior of cascaded rectangular-triangular fin is analytically and numerically investigated using MATLAB. Three different methods of solution are implemented in this study. First, an analytical closed form solution of the problem under-consideration is obtained manually since the governing equations and their boundary conditions are linear. Second, the solution is developed in symbolic form using MATLAB symbolic toolbox and compared with developed closed form analytical solution. Third, a two-dimensional form of the problem at hand is solved using finite elements method via MATLAB's PDE (partial differential equation) toolbox and compared with analytical solution. The results of this study show that MATLAB can be used effectively and efficiently to solve challenging heat transfer problems.

**Keywords:** MATLAB, Symbolic solution, Analytical, Finite element, Cascaded fin, Thermal behavior.

### 1. INTRODUCTION

Finned surfaces (extended surface) are commonly used in practice to enhance heat transfer rate between a solid and an adjoining fluids. Disposition of fins on the base surface results in increase of the total area of heat transfer which in turn increase the heat transfer rate. Different fins configuration are presented and analyzed in ref. [1].

Typical application of the fins are originated in the cooling devices for electronic equipment and compact heat exchangers that used in many application such as air conditions, car radiators, etc...[1]. A comprehensive review of fin surface technology has been provided in a book by Kraus, Aziz and Welty [1].

In the typical analysis of fins, the steady operation with no heat generation, constant thermal conductivity and uniform convection heat transfer coefficient are considered. In the analysis of fin geometries, attention must give to constraints or assumptions that employed to defin and limit the problem and often to simplify its solutions. Details analyses of a wide variety of find surface geometries were discussed in ref. [1].

Finned array can be constructed by using a single fin as building blocks as shown in Figure 1. Typical example is the heat sinks (special designed finned surfaces), that involve one-of-a-kind complex geometries, which are extensively used in the design of cooling device of electronic equipment. A comprehensive treatment of these finned arrays is provided by ref. [1]. The analysis and design information available for a single fin can be readily adapted to design the assembly and/or predict the performance.

The solutions of the single fin such as triangular or rectangular fin as shown in Figure 1a, 1b involves only one ordinary differential equation with two boundary condition, however in the case of cascaded fin as shown in Figure 2 the analysis involves the solution of a set of coupled, ordinary differential equations with two boundary conditions associated with each differential equation. In case of the linear differential equations and the boundary conditions, the solution procedure is conceptually simple but algebraically very tedious.

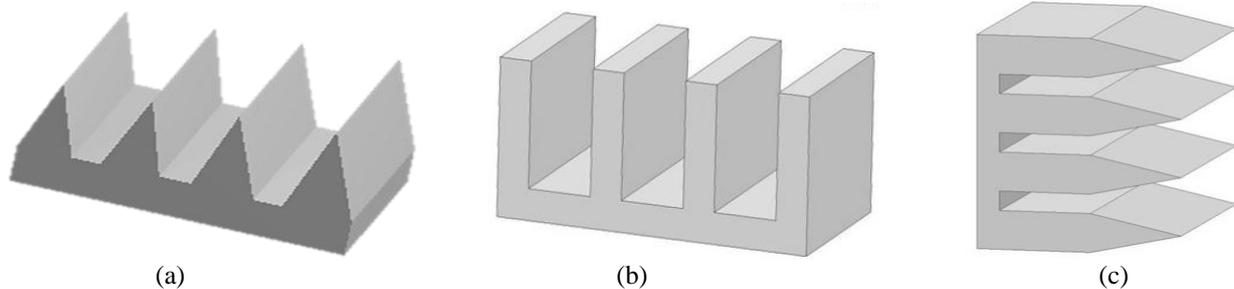


Figure 1. Examples of array extended surface: (a) triangular assembly (b) rectangular assembly (c) cascaded rectangular-triangular

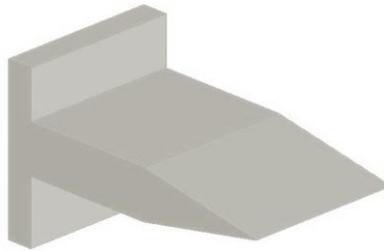


Figure 2. The cascaded rectangular-triangular fin

During last decades, computational software's (CS) are extensively used to solve the heat transfer problem. Recently, computational software's (CS) such as MATLAB, Maple, and Mathematica is growing up very rapidly. In fact, Computational software is playing a major role in understanding engineering mathematics problems. The CS characterized by its potential power of solving engineering mathematical problems in both symbolic and numerical computation. CS symbolic toolbox provide alternative to hand analysis which results in easy generation and manipulation of solutions. Aziz [2] provided a comprehensive analysis of a variety of heat transfer problems such as finned surface model using MAPLE software. In addition, Aziz [3] investigated the performance analysis of a cascaded rectangular-triangular fin using MAPLE.

Furthermore, some new solutions for extended surface heat transfer which not exist in open literature were developed by Aziz, and Mcfadden [4] using Maple. MATLAB, is one of the most popular CS packages and is used extensively in solving engineering problems. MATLAB is characterized by its capability of both symbolic and numerical computations. MATLAB is used to solve the problem at hand due to its a graphics package in 2-D and 3-D graphics; its simple programming language; its potential power of solving mathematics problems in both symbolic and numerical computation, its large families of application toolboxes; its power-ful matrix laboratory, which leads to practice experimental mathematics in an easy feedback loop; and its great number of books and user manuals. The main source to learn about MATLAB is through the manuals that come with software. In addition, an excellent learning resources for MATLAB is the Mathworks website [5], learning guides for Matlab such as [6,7,8,9], and numerical methods books using MATLAB such as [10,11].

Extensive literature review on the solution of cascaded fin assembly shows that no attempt was done to solve this problem using MATLAB. Therefore, the main objective of this research work is to illustrate the effectiveness of using MATLAB in solving such problems. In this paper the thermal behavior of cascaded fin assembly model is theoretically investigated using three different methods of solution. First, closed form solution developed using hand analysis since the coupled differential equations and the boundary conditions are linear. Second, the solution is developed in sym-bolic form using MATLAB symbolic toolbox and compared with developed closed form analytical solution. Third, finite elements are used to construct a two-dimensional numerical solution for the fin mode using MATLAB's PDE (partial differential equation) toolbox. Excellent agreement between the results obtained using these three different methods were found.

## 2. MATHEMATICAL MODEL

The extended surface model under investigation is a Cascaded Rectangular-Triangular Fin as shown in Fig. 3. It is triangular fin of length  $L_1$  in cascade with a rectangular fin such that the overall length of assembly is  $L$ . The base height is  $w$ . The width of the assembly is assumed to be  $b$  (the dimension that is perpendicular to the page) which is  $b \gg w$  large. For mathematical convenience, the origin of the coordinate ( $x$ ) is located at the assembly's tip. The assembly's base ( $x = L$ ) is maintained at a constant, uniform temperature  $T_b$ . The thermal conductivity for assembly is  $k$ . The assembly interacts thermally with an ambient at  $T_\infty$  through a heat transfer coefficient  $h$ . The parameters  $k$  and  $h$  are assumed to be constant. For the formulation of the model, the parameters and coordinate system for assembly are illustrated in Figure 3.

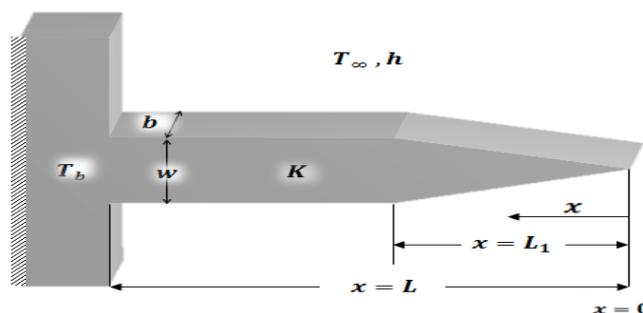


Figure 3. Cascaded Rectangular-Triangular fin model

## 2.1 Mathematical Formulations

The mathematical formulation of the temperature distribution for the fin assembly is based on the energy balance for an elemental volume of triangular and rectangular fins. The governing differential equation for the temperature distribution in the triangular fin is given by:

$$\frac{\partial T_1^2}{\partial x^2} + \frac{1}{x} \frac{\partial T_1}{\partial x} - P^2 \frac{T_1}{x} = 0 \quad 0 \leq x \leq L_1 \quad (1a)$$

where:  $p = \sqrt{\frac{2fhL_1}{Kw}}$ ,  $f \cong 1$

The boundary conditions are:

at :  $x = 0$   $T_1(x) = T_0$  (2a)

at :  $x = L_1$   $T_1(x) = T_j$  (3a)

where  $T_0$  is the fin's tip temperature (finite value) and  $T_j$  is temperature excess at junction between rectangular and triangular fins. All other parameters have been defined in the nomenclature.

To find the interface temperature  $T_j$ , the continuity of heat flux is used at interface:

$$-K \left( \frac{\partial T_1}{\partial x} \right)_{x=L_1} = -K \left( \frac{\partial T_2}{\partial x} \right)_{x=L_1} \quad (4a)$$

Similarly, the governing differential equation for the temperature distribution in the rectangular fin is given by:

$$\frac{\partial^2 T_2}{\partial x^2} - m^2(T_2 - T_\infty) = 0 \quad L_1 \leq x \leq L \quad (1b)$$

where :  $m = \sqrt{\frac{2h}{kw}}$

The boundary conditions are:

at :  $x = L_1$   $T_2(x) = T_j$ , (2b)

$$-K \left( \frac{\partial T_1}{\partial x} \right)_{x=L_1} = -K \left( \frac{\partial T_2}{\partial x} \right)_{x=L_1} \quad (3b)$$

at :  $x = L$   $T_2(x) = T_b$  (4b)

where all other parameters are defined in the nomenclature.

## 2.2 Dimensionless Analysis

To get more general representation of the fin's model, the following dimensionless quantities are introduced:

$$\theta_1 = \frac{T_1(x) - T_\infty}{T_b - T_\infty} \quad \theta_2 = \frac{T_2(x) - T_\infty}{T_b - T_\infty} \quad \theta = \frac{T(x) - T_\infty}{T_b - T_\infty}$$

$$m^{+2} = \frac{2hL^2}{kw} \quad x^+ = \frac{x}{L} \quad P^{+2} = \frac{2hL_1L}{kw} \quad x_1^+ = \frac{L_1}{L}$$

in terms of these non-dimensional variables the equations of each fin's can be written as:

## Triangular Fin

The differential dimensionless equation for the rectangular fin may be written as:

$$\frac{\partial^2 \theta_1}{\partial x^{+2}} + \frac{1}{x^+} \frac{\partial \theta_1}{\partial x^+} - P^{+2} \frac{\theta_1}{x^+} = 0 \quad 0 \leq x^+ \leq x_1^+ \quad (3a)$$

at :  $x^+ = 0$   $\theta_1(x^+) = \frac{T_0 - T_\infty}{T_b - T_\infty} = \theta_0$  (3b)

at :  $x^+ = x_1^+$   $\theta_1(x^+) = \frac{T_j - T_\infty}{T_b - T_\infty} = \theta_j$  (3c)

and :  $-K \left( \frac{\partial \theta_1}{\partial x^+} \right)_{x^+=x_1^+} = -K \left( \frac{\partial \theta_2}{\partial x^+} \right)_{x^+=x_1^+}$  (3d)

## Rectangular Fin

Similarly the differential dimensionless equation for the triangular fin may be written as:

$$\frac{\partial^2 \theta_2}{\partial x^{+2}} - m^{+2} \theta_2(x^+) = 0 \quad x_1^+ \leq x^+ \leq 1 \quad (4a)$$

at :  $x^+ = x_1^+ \rightarrow \theta_2(x^+) = \frac{T_j - T_\infty}{T_b - T_\infty} = \theta_j$  (4b)

and  $-K \left( \frac{\partial \theta_1}{\partial x^+} \right)_{x^+=x_1^+} = -K \left( \frac{\partial \theta_2}{\partial x^+} \right)_{x^+=x_1^+}$  (4c)

at :  $x^+ = 1 \rightarrow \theta_2(x^+) = \frac{T_b - T_\infty}{T_b - T_\infty} = \theta_b = 1$  (4d)

The fin heat transfer rate,  $q$ , may be found by evaluating the conduction heat transfer at  $x = L$  or  $x^+ = 1$

$$q = -kA \left. \frac{\partial T_2(x)}{\partial x} \right|_{x=L} \quad (5)$$

Or, in dimensionless form:

$$q^+ = \frac{qL}{kA(T_b - T_\infty)} = \left. \frac{\partial \theta_2(x^+)}{\partial x^+} \right|_{x^+=1} \quad (6)$$

The fin base temperature,  $\theta_b$ , can be obtained by substitution  $\theta^+ = 1$  in the solution of  $\theta_2(x^+)$ , i.e.

$$\theta_b = \theta_2(x^+) |_{x^+=1} \quad (7)$$

## 3. ANALYTICAL SOLUTION

### 3.1 Triangular Fin

The general solution of problem set (3) is:

$$\theta_1^+ = A_1 I_0 \left( 2P^+ x^{+\frac{1}{2}} \right) + A_2 K_0 \left( 2P^+ x^{+\frac{1}{2}} \right) \quad (8)$$

Constants  $A_1$  &  $A_2$  are determined by applying boundary conditions (3b) and (3c) at the two end of fin. One can note that, the constant  $A_2$  must be zero, since the Bessel function  $k_0$  is infinite for zero argument. So, we get:

$$\theta_1^+ = A_1 I_0 \left( 2 P^+ x^{+\frac{1}{2}} \right) \quad (9)$$

To find  $A_1$ , we use the boundary condition at  $x^+ = x_1^+$ , i.e.,  $\theta_1(x_1^+) = \theta_j$

$$A_1 = \frac{\theta_j}{I_0 \left( 2 P^+ \sqrt{x_1^+} \right)} \quad (10)$$

Finally the solution becomes:

$$\theta_1(x^+) = \frac{\theta_j}{I_0 \left( 2 P^+ \sqrt{x_1^+} \right)} \cdot I_0 \left( 2 P^+ \sqrt{x^+} \right) \quad (11)$$

Which involves unknown  $\theta_j$  to be determined from continuity of heat flux at the interface after finding the rectangular fin solution.

### 3.2 Rectangular fin

The general solution of problem set (4) is:

$$\theta_2(x^+) = \beta_1 \sinh(mx^+) + \beta_2 \cosh(mx^+) \quad (12)$$

Apply the boundary conditions:

$$\text{At: } x^+ = x_1^+ \quad \theta_2(x^+) = \theta_j$$

$$\theta_2(x_1^+) = \beta_1 \sinh(mx_1^+) + \beta_2 \cosh(mx_1^+) = \theta_j \quad (13)$$

$$\text{At: } x^+ = 1 \quad \theta_2(x^+) = 1$$

$$\theta_2(1) = \beta_1 \sinh(m) + \beta_2 \cosh(m) = 1 \quad (14)$$

Solve equations (13) & (14) for constants  $\beta_1$  and  $\beta_2$ :

$$\beta_2 = \frac{c_1 - \theta_j c_2}{c_3} \quad (15)$$

$$\frac{\partial \theta_2}{\partial x^+} = \left( \theta_j C_4 - \frac{c_1 - \theta_j c_2}{c_3} \cdot C_5 \right) m \cdot \cosh(mx^+) + \left( \frac{c_1 - \theta_j c_2}{c_3} \right) \cdot m \sinh(mx^+) \quad (24)$$

$$\frac{\partial \theta_2}{\partial x^+} \Big|_{x^+=x_1^+} = \left( \theta_j C_4 - \frac{c_1 - \theta_j c_2}{c_3} \right) \cdot m \cosh(mx^+) + \left( \frac{c_1 - \theta_j c_2}{c_3} \right) \cdot m \sinh(mx^+) \quad (25)$$

Substitute equation (23) and (25) into equation (19), one can get:

$$\theta_j = \frac{C_9 - C_7}{C_0 - C_6 - C_8 + C_{10}} \quad (26)$$

Where

$$C_0 = \frac{p}{\sqrt{x_1^+}} \cdot \frac{I_1(2p\sqrt{x_1^+})}{I_0(2p\sqrt{x_1^+})}, \quad C_1 = \sinh(mx_1^+), \quad C_2 = \sinh(m)$$

$$C_3 = \sinh(mx_1^+) \cdot \cosh(m) - \cosh(mx_1^+) \cdot \sinh(m)$$

$$\beta_1 = \theta_j C_4 - \frac{c_1 - \theta_j c_2}{c_3} \cdot C_5 \quad (16)$$

Where

$$C_1 = \sinh(mx_1^+), \quad C_2 = \sinh(m) \quad (17a)$$

$$C_3 = \sinh(mx_1^+) \cdot \cosh(m) - \cosh(mx_1^+) \cdot \sinh(m) \quad (17b)$$

$$C_4 = \frac{1}{\sinh(mx_1^+)}, \quad C_5 = \frac{\cosh(mx_1^+)}{\sinh(mx_1^+)} \quad (17c)$$

Finally the solution becomes

$$\theta_2(x^+) = \left[ \theta_j C_4 - \frac{c_1 - \theta_j c_2}{c_3} C_5 \right] \sinh(mx^+) + \left[ \frac{c_1 - \theta_j c_2}{c_3} \right] \cdot \cosh(mx^+) \quad (18)$$

### 3.3 Interface Temperature

As one can see, the solution of both fins involves unknown  $\theta_j$  which can be found from continuity of heat flux at the interface as follows

$$\frac{d\theta_1(x^+)}{dx^+} \Big|_{x^+=x_1^+} = \frac{d\theta_2(x^+)}{dx^+} \Big|_{x^+=x_1^+} \quad (19)$$

Where

$$q_1^+ = \frac{q_1 L}{k t w (T_b - T_\infty)} = \frac{d\theta_1(x^+)}{dx^+} \Big|_{x^+=x_1^+} \quad (20)$$

$$q_2^+ = \frac{q_2 L}{k t w (T_b - T_\infty)} = \frac{d\theta_2(x^+)}{dx^+} \Big|_{x^+=x_1^+} \quad (21)$$

Next step is to find the  $\frac{d\theta_1}{dx^+} \Big|_{x_1^+}$  and  $\frac{d\theta_2}{dx^+} \Big|_{x_1^+}$

$$\frac{\partial \theta_1}{\partial x^+} = \frac{\theta_j}{I_0(2p\sqrt{x_1^+})} \cdot \frac{2p}{2\sqrt{x^+}} I_1(2p\sqrt{x^+}) \quad (22)$$

$$\frac{\partial \theta_1}{\partial x^+} \Big|_{x^+=x_1^+} = \frac{\theta_j}{I_0(2p\sqrt{x_1^+})} \cdot \frac{p}{\sqrt{x^+}} I_1(2p\sqrt{x_1^+}) = \theta_j \cdot C_0 \quad (23)$$

$$C_4 = \frac{1}{\sinh(mx_1^+)} \quad , \quad C_5 = \frac{\cosh(mx_1^+)}{\sinh(mx_1^+)} \quad , \quad C_6 = C_4 \cdot m \cdot \cosh(mx_1^+)$$

$$C_7 = \frac{C_1 C_5}{C_3} \cdot m \cdot \cosh(mx_1^+) \quad , \quad C_8 = \frac{C_2 C_5}{C_3} \cdot m \cdot \cosh(mx_1^+)$$

$$C_9 = \frac{C_1}{C_3} \cdot m \cdot \sinh(mx_1^+) \quad , \quad C_{10} = \frac{C_2}{C_3} \cdot m \cdot \sinh(mx_1^+)$$

### 3.4 Solution of Cascaded Fin

Finally, the solution of the problem becomes:

$$\theta_1(x^+) = \frac{\theta_j}{I_0(2P^+\sqrt{x^+})} \cdot I_0(2P^+\sqrt{x^+}) \tag{27}$$

$$\theta_2(x^+) = \left[ \theta_j C_4 - \frac{C_1 - \theta_j C_2}{C_3} C_5 \right] * \sinh(mx^+) + \left[ \frac{C_1 - \theta_j C_2}{C_3} \right] * \cosh(mx^+) \tag{28}$$

Where:

$$\theta_j = \frac{C_9 - C_7}{C_0 - C_6 - C_8 + C_{10}}$$

$$C_0 = \frac{p}{\sqrt{x_1^+}} \cdot \frac{I_1(2p\sqrt{x_1^+})}{I_0(2p\sqrt{x_1^+})} \quad , \quad C_1 = \sinh(mx_1^+) \quad , \quad C_2 = \sinh(m)$$

$$C_3 = \sinh(mx_1^+) \cdot \cosh(m) - \cosh(mx_1^+) \cdot \sinh(m)$$

$$C_4 = \frac{1}{\sinh(mx_1^+)} \quad , \quad C_5 = \frac{\cosh(mx_1^+)}{\sinh(mx_1^+)} \quad , \quad C_6 = C_4 \cdot m \cdot \cosh(mx_1^+)$$

$$C_7 = \frac{C_1 C_5}{C_3} \cdot m \cdot \cosh(mx_1^+) \quad , \quad C_8 = \frac{C_2 C_5}{C_3} \cdot m \cdot \cosh(mx_1^+)$$

$$C_9 = \frac{C_1}{C_3} \cdot m \cdot \sinh(mx_1^+) \quad , \quad C_{10} = \frac{C_2}{C_3} \cdot m \cdot \sinh(mx_1^+) \tag{29}$$

### 3.5 Alternative Solution (Superposition)

The geometry of the model under investigation is a combination of a straight-rectangular and a straight-triangular fin, with the excess temperature  $\theta_j$  and heat flow  $q_j$  at junction between the two fins, as shown in Figure 4.

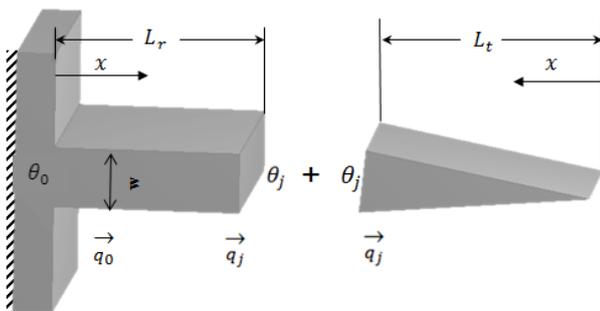


Figure 4. Combination of rectangular and triangular fins.

The solution for each fin can be formed separately and using the continuity of heat flux at the interface to find the

excess temperature  $\theta_j$ . The solution of rectangular fin, with  $\theta_0$  at one end and  $\theta_j$  at other, is:

$$\theta(x) = \frac{\theta_0 \sinh(m(L_r - x)) + \theta_j \sinh(mx)}{\sinh(mL_r)} \tag{30}$$

Using Fourier's law with equation (30), the rate of heat flow at the two ends of the fin is:

$$q_0 = kmw \frac{\theta_0 \cosh(mL_r) - \theta_j}{\sinh(mL_r)} \tag{31}$$

$$q_j = kmw \frac{\theta_0 - \theta_j \cosh(mL_r)}{\sinh(mL_r)} \tag{32}$$

Where:  $= 1xw = w$ , and  $m = \sqrt{\frac{2h}{kw}}$

The solution of the triangular fin is:

$$\theta(x) = \theta_j \frac{I_0(2px^{1/2})}{I_0(2pL_t^{1/2})} \tag{33}$$

Again using Fourier's law with equation (33), the heat flow at  $L_t$  is:

$$q_j = -kw\theta_j \frac{P}{\sqrt{L_t}} \cdot \frac{I_1(2P\sqrt{L_t})}{I_0(2P\sqrt{L_t})} \quad (34)$$

due to change in coordinate direction for triangular fin, the minus sign should be added to equation (34)

$$-q_j = -kw\theta_j \frac{P}{\sqrt{L_t}} \cdot \frac{I_1(2P\sqrt{L_t})}{I_0(2P\sqrt{L_t})} \quad (35)$$

Where:

$$p = \sqrt{\frac{2hL_t}{Kw}} = \sqrt{\frac{2h}{Kw}} \cdot \sqrt{L_t} = m \cdot \sqrt{L_t}$$

Equate equation (32) and (35), and solve for  $\theta_j$  to get

$$\theta_j = \frac{\theta_0}{\cos h(mL_r) + \sin h(mL_r) \frac{I_1(2mL_t)}{I_0(2mL_t)}} \quad (36)$$

Knowing  $\theta_j$ , one can get the temperature distribution and heat flow for the fin assembly.

#### 4. SYMBOLIC SOLUTION OF CASCADED FIN USING MATLAB®

The MATLAB commands used to find the symbolic solution of the cascaded fin model are shown in the Table

Table 1. MATLAB Commands

Command	Description
Clc	Clear Command Window
Clear	Remove items from workspace, freeing up system memory
Disp	Display text or array
Dsolve	Symbolic solution of ordinary differential equations
Solve	Symbolic solution of algebraic equations
Pretty	Pretty print symbolic expressions
Subs	Symbolic substitution in a symbolic expression or matrix
Diff	Differences and approximate derivatives
Syms	Short-cut for constructing symbolic objects
Fprintf	Write formatted data to file
Simplify	Symbolic simplification
Sqrt	Square root
Eval	Execute a string containing a MATLAB expression
diary filename	copies everything that subsequently appears in the Command Window to the text file filename
Diary off	Stop recording the session

To get numerical values for temperatures and heat rate, the model data should feed to the MATLAB command window or to scrip m-file. Its highly recommend to save the work in scrip m-file for easy manipulation and recording. If command window is used, there is another way of capturing output with the diary command. The command diary filename copies everything that subsequently appears in the Command Window to the text

(1). The MATLAB program used to obtain the symbolic solution of the problem at hand is presented in the appendix. To get numerical values for temperatures and heat rate, the model data should feed to the MATLAB command window or to scrip m-file. Its highly recommend to save the work in scrip m-file for easy manipulation and recording. If command window is used, there is another way of capturing output with the diary command. The command diary filename copies everything that subsequently appears in the Command Window to the text file filename. You can then edit the resulting file with any text editor (including the MATLAB Edi-tor). The command for stop recording the session is Same model can be solved using the MUPAD note-book interface in the symbolic math toolbox. To accesses the mupad editor just type mupad in the MATLAB command window. Figure 5 shows a snap shot of mu-pad program file that used to solve the cascaded fin model.

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file filename. You can then edit the resulting file with any text editor (including the MATLAB Editor). The command for stop recording the session is diary off.

Same model can be solved using the MUPAD notebook interface in the symbolic math toolbox. To accesses the mupad editor just type mupad in the MATLAB command

window. Figure 5 shows a snap shot of mupad program

file that used to solve the cascaded fin model.

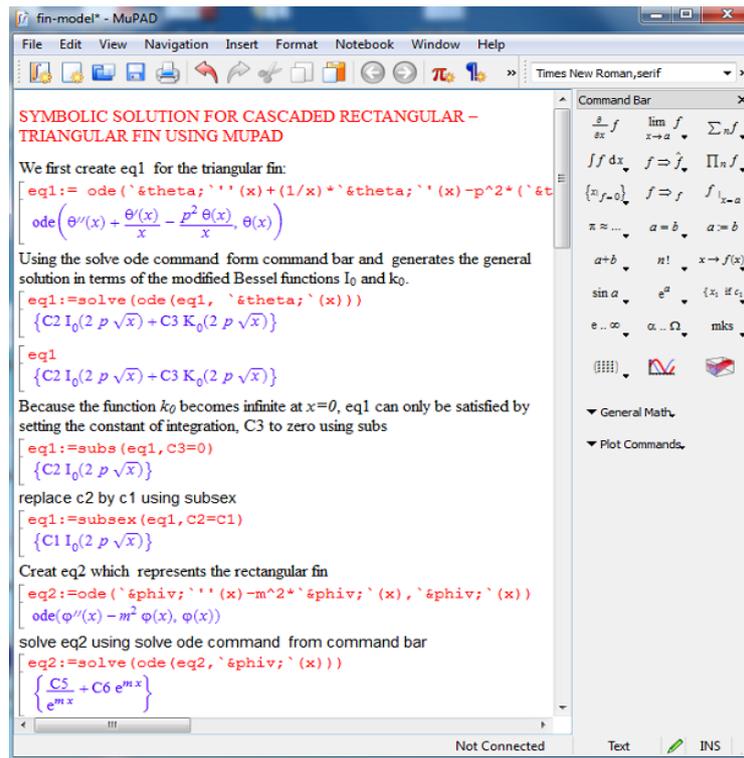


Figure 5 A snap shot of mupad notebook showing derivation of symbolic solution of cascaded fin model.

## 5. RESULTS

### 5.1 Interface Temperature and Heat Flux

The solution for the fin interface temperature  $T_j$  can be obtained as follows:

```
>>k=85.5;h=34;w=2/1000;T_b=150;T_inf=19;L=(5+5)/10
0; L1=5/100;X1=L1/L; % data;
```

```
>>Bi=(h*w)/k;p=sqrt((2*h*L*L1)/(k*w));m=sqrt((2*h*L
^2)/(k*w));
```

```
>> x=X1; % at the fin interface
```

```
>>theta_1=eval(theta_1) % using theta_1
```

```
>>theta_2=eval(theta_2) % using theta_2 to check
```

```
>> Tj=theta_1*(T_b-T_inf)+T_inf
```

Also, the heat flux can be found using the following commands

```
>> x=1; % at the base
```

```
>> q=eval(q2)
```

Table 2 shows the comparisons between the interface temperature obtained by the closed form solutions and the MATLAB symbolic solutions. It is clear that there is an excellent agreement between the two solutions.

Table 2. Analytical and symbolic results for assembly model with  $x_1^+ = 0.5$

	$\theta_j[-]$	$T_j$ °C	$q_2[-]$	$q_b$ W/m <sup>2</sup>
Analytical solutions	0.8735	133.4	0.3464	785
Symbolic solutions	0.8735	133.4	0.3464	785

### 5.2 Dimensionless Temperatures Distribution

Figure 6 shows the dimensionless distribution for a cascaded rectangular-triangular fin for a different length ratio ( $x_1^+ = L_1/L$ ). The temperature distribution increasing with decreasing the length ratio. Figure 7 shows the variation of the interface temperature  $\theta_j$  with length ratio ( $x_1^+ = L_1/L$ ). It can be seen that the interface temperature significantly increases as the length ratio increases.

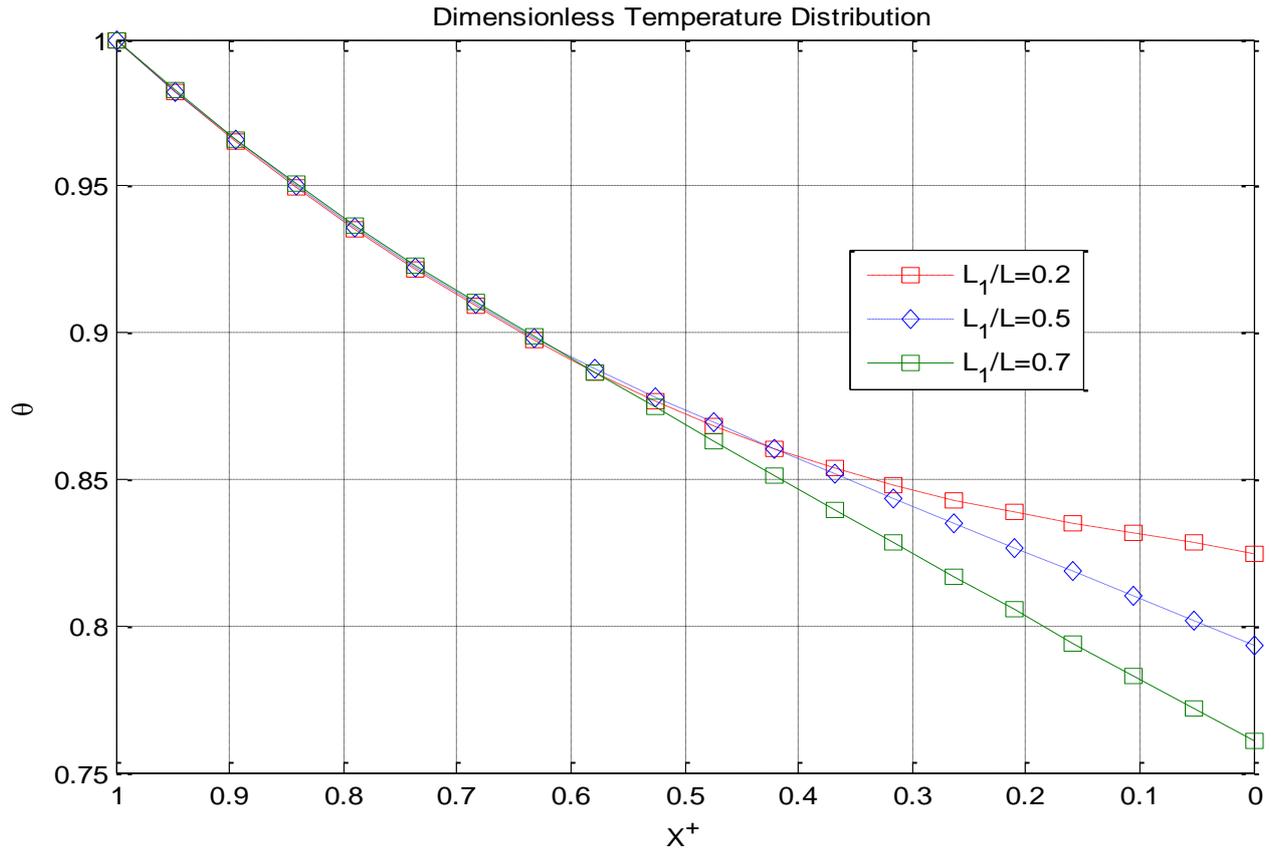


Figure 6. Dimensionless temperatures distribution for assembly different length ration

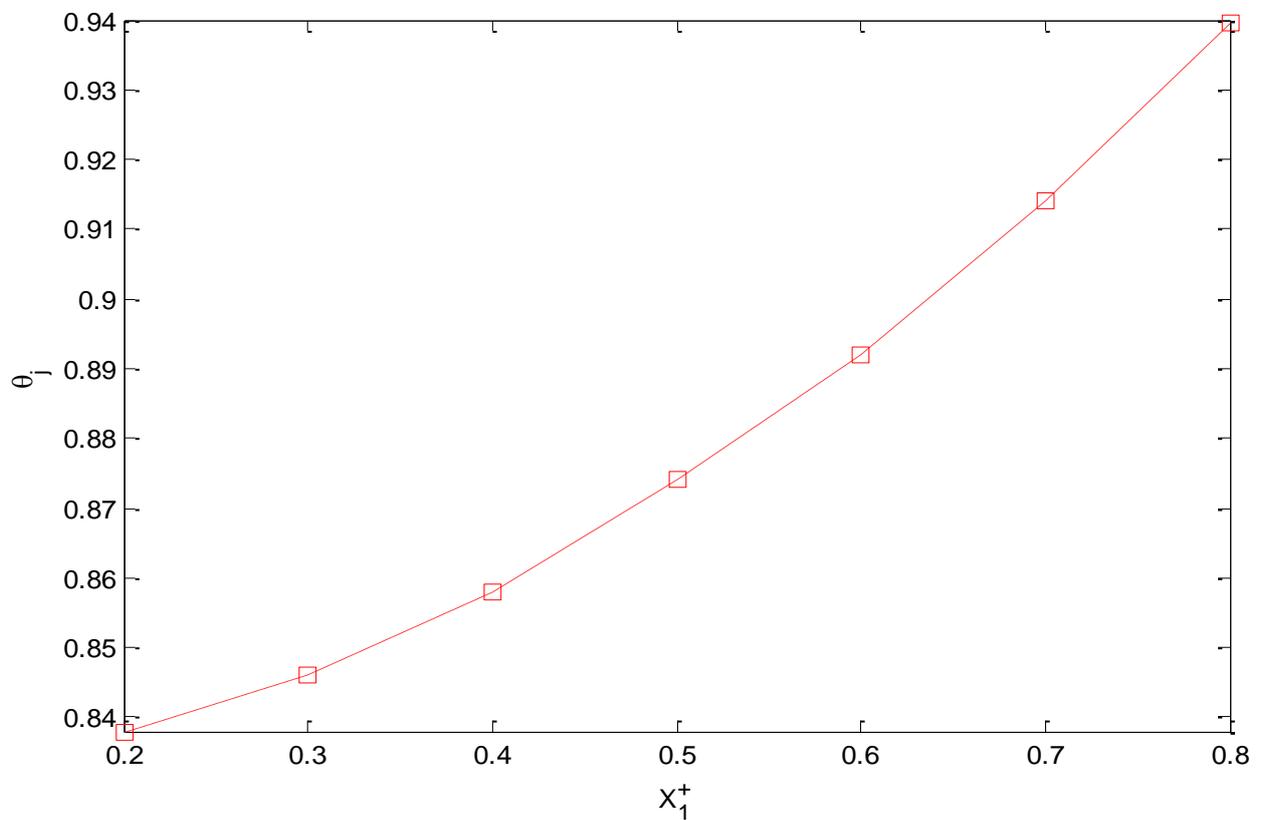


Figure 7 Interface dimensionless temperatures  $\theta_j$  versus length ratio  $x_1^+$

## 6. TWO DIMENSIONAL SOLUTION

The mathematical model of the cascaded fin assembly can be described by the heat conduction equation with constant properties. To be consistent with one non-dimensional

analysis, the non-dimensional Laplace equation with boundary conditions presented as follows: on region D as shown in the Figure 8.

$$\frac{\partial^2 \theta}{\partial x^{*2}} + \frac{\partial^2 \theta}{\partial y^{*2}} = 0, \quad (37)$$

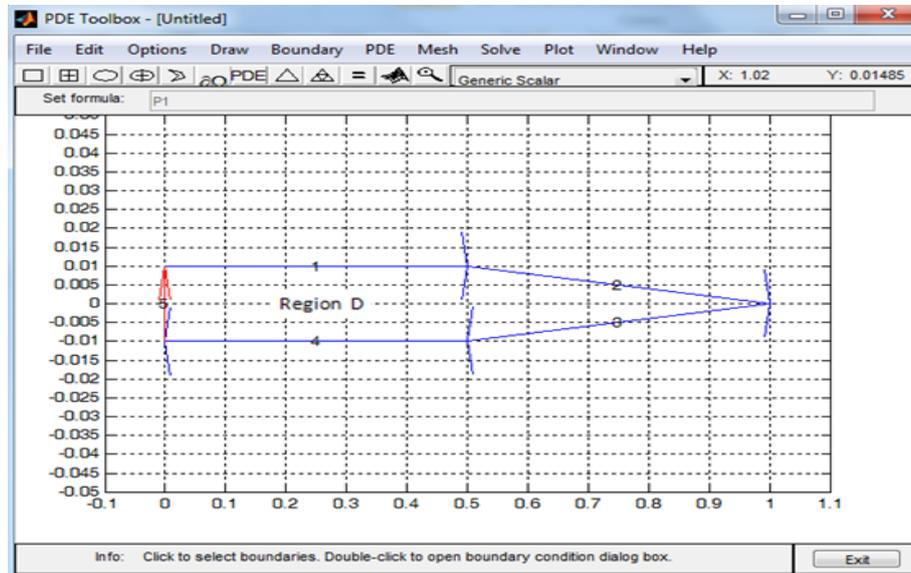


Figure 8. Domain and Boundary conditions for Laplace equation

With following boundary conditions:

$$-(\hat{n} \cdot \nabla \theta)_{y=\mp \frac{w}{2}} = Bi_L \theta \left( x, \mp \frac{w}{2} \right), \text{ for } (x^+, y^+) \in L_1 \text{ and } (x^+, y^+) \in L_4 \quad (38)$$

$$-(\hat{n} \cdot \nabla \theta)_{y=\mp \frac{w/2}{L_1}} = Bi_L \theta \left( x, \mp \frac{w/2}{L_1} x \right), \text{ for } (x^+, y^+) \in L_2 \text{ and } (x^+, y^+) \in L_3 \quad (39)$$

$$\theta(1, y^+) = 1, \text{ for } (x^+, y^+) \in L_5 \quad (40)$$

Using the MATLAB's PDE toolbox, the finite element solution for above model can be generated. Figure 9 depicts the temperature distribution as the surface of  $\theta(x, y)$  for the fin assembly with different length ratio ( $x_1^+ = L_1/L = 0.5$ ).

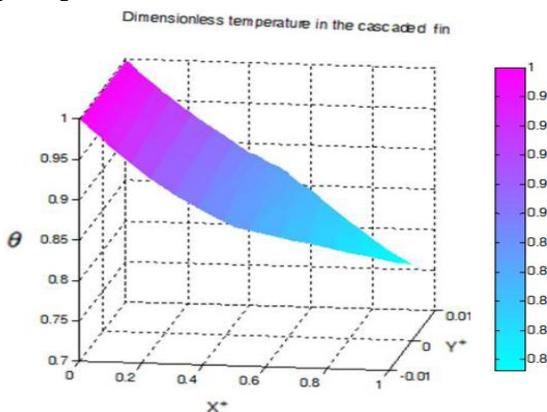


Figure 9. Dimensionless temperatures distribution in the cascaded fin.

To manipulate and analysis the data generate in the figure 9, PDE toolbox allows the user to export the computational results to MATLAB workspace and stored it in model variable names. To find the temperature  $\theta(x^+, 0)$  to compare with one-dimensional analysis the following MATLAB commands are used:

```
>> x=0:0.1;
>> T_at_centar_line=tri2grid(p,t,u,x,y) %computes the
function values T over the grid
%defind by the vectors x and y, from the function u with
%values on the triangular mesh defind by p and t ( in this
%case y=0).
```

Figure 10 shows a snap shot of MATLAB command window which include the command used to find temperatures at  $y=0$  (the assembly axis of symmetry) and the interface temperature. The results are compared with 1-D solution in Figure 11. The one-dimensional results and the two-dimensional calculation are excellent agreement.

```

Command Window
New to MATLAB? Watch this Video, see Demos, or read Getting Started.

>> x=0:0.1:1; %define a vector x
>> T_at_center_line=tri2grid(p,t,u,x,0);
>> %computes the function values T over the grid defined by the vectors x and y=0
>> %from the function u with values on the triangular mesh defined by p and t
>> T_at_center_line=tri2grid(p,t,u,x,0)

T_at_center_line =

    1.0000    0.9668    0.9375    0.9119    0.8900    0.8716    0.8550    0.8387    0.8225    0.8064    0.7905

>> %T at the interface
>> T_at_interface=tri2grid(p,t,u,0.5,0)

T_at_interface =

    0.8716

fx >> |
Workspace
Name ^ Value
T_at_center_line <1x11 double>
T_at_interface 0.8716
e <7x156 double>
p <2x375 double>
t <4x592 double>
u <375x1 double>
x <1x11 double>
    
```

Figure 10. Snap shot of MATLAB command window shows method of finding the temperature at  $y=0$  (the assembly axis of symmetry) to compare with -1D solution

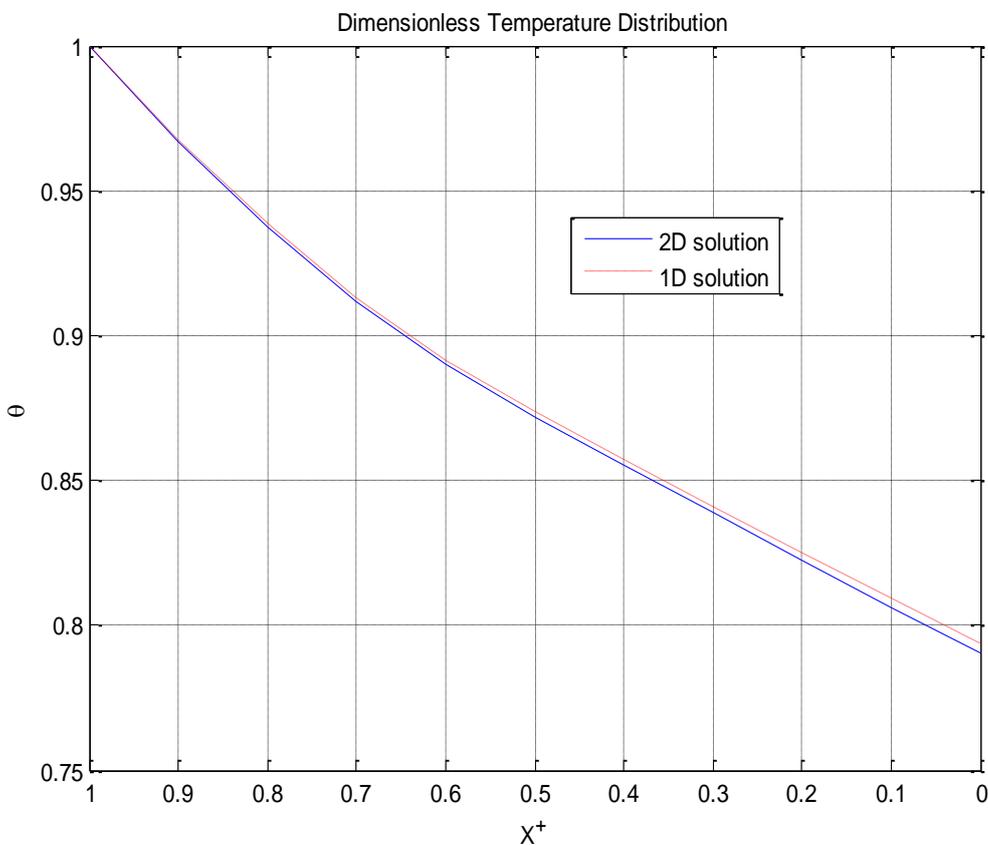


Figure 11. Comparison of dimensionless temperatures of 1-D and 2-D solution at the assembly axis of symmetry.

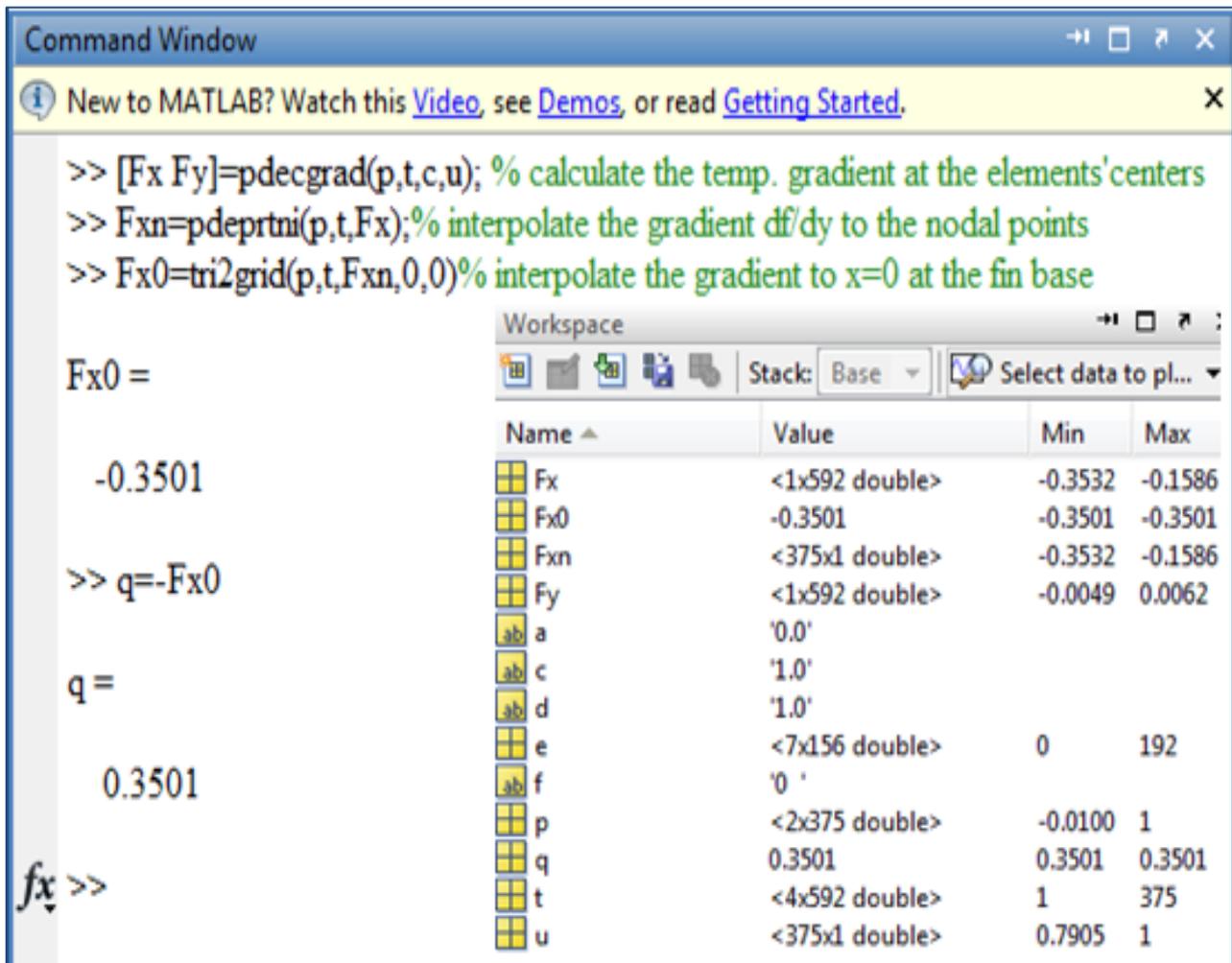


Figure 12 Snap shot of MATLAB command window shows method of finding the heatflux at the assembly base using 2-D solution.

Also, the heat rat at the base of the fin can be calculated using MATLAB builtin functions as shown in Figure 12. As on can see, the value of the heat flux is in good agreement with value from one dimensional analysis.

## 7. CONCLUSIONS

MATLAB includes several powerful toolboxes to solve most common engineering problems. In this paper, MATLAB has been proved to be an effective and convenient tool for developing symbolic and numerical solutions of a cascaded fin assembly. MATAALB solutions were compared with developed hand solution and an excellent agreement was found. The use of MATLAB has also provided an excellent way to investigate the model performance and show the results in form of graphs or numerical values. Hard work and time consuming of developing the analytical solution for the most common engineering problems can be eliminated by using symbolic toolbox in MATAALB. With confident, MATLAB can be used efficiently to solve challenging heat transfer problems.

## NOMENCLATURE

T	temperature
x	axial coordinate
L	cascaded fin length
$L_1$	triangular fin length
w	base height
b	width of the assembly
Bi	Biot number
k	thermal conductivity
h	convection heat transfer coefficient
P	triangular fin performance factor
m	rectangular fin performance factor
$T_b$	base temperature
$T_j$	interface temperature
$\theta$	dimensionless temperature
$\theta_1$	dimensionless temperature of triangular fin
$\theta_2$	dimensionless temperature of rectangular fin
$\theta_b$	dimensionless temperature at base
$\theta_j$	dimensionless temperature at interface
$\theta_0$	dimensionless temperature at tip

$x^+$	dimensionless axial coordinate
$x_1^+$	dimensionless length ratio
$P^+$	dimensionless triangular fin performance factor
$m^+$	dimensionless rectangular fin performance factor
A	cross-sectional area of the fin
q	fin heat transfer rate
$q_0$	fin heat transfer rate at base
$q_j$	fin heat transfer rate at fin junction
$q^+$	dimensionless heat transfer rate

## SUBSCRIPTS

1, t	triangular fin
2, r	rectangular fin
b, 0	base
$\infty$	ambient
j	junction

## SUPERSCRIPTS

+	dimensionless
---	---------------

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## APPENDIX

### Symbolic Solution Method

First eq. (3a) for the triangular fin must be created in MATLAB<sup>®</sup> and say call it Eq1. MATLAB commend is:

```
>> Eq1='D2theta1+(1/x)*Dtheta1-p^2*(theta1/x)=0'
```

Next Eq1 is solved using the dsolve command and MATLAB<sup>®</sup> generates the general solution in terms of the modified Bessel functions  $I_0$  and  $k_0$ . MATLAB command is

```
>> theta_1=dsolve(Eq1,'x');
```

Because the function  $k_0$  becomes infinite at  $x=0$ , eq.(4.3) can only be satisfied by setting the constant of integration,  $C_2$  to zero. MATLAB command is

```
>> theta_1=subs(theta_1,'C2',0);
>> theta_1=subs(theta_1,'C1',sym('c1'));
```

Equation (4a) which represents the rectangular fin may now be created. MATLAB commend is:

```
>> Eq2='D2theta2-m^2*theta2=0'
```

and solved using the dsolve command. MATLAB commend is:

```
>> theta_2=dsolve(Eq2,'x');
```

Because the constants  $C_1$  and  $C_2$  appear in the solution for  $\theta_1(x)$ , we replace them with  $C_3$  and  $C_4$  to avoid confusion. MATLAB commends are:

```
>> syms C1 C2
>> theta_2=subs(theta_2,{C1,C2},{sym('c3'),sym('c4')});
```

The derivative of  $\theta_1(x)$  is obtained to be used in boundary conditions (3d and 4c). MATLAB commend is:

```
>> q1=diff(theta_1,'x');
```

The derivative of  $\theta_2(x)$  is obtained to be used in boundary conditions (3d and 4c) end equation (6). MATLAB commend is:

```
>> q2=diff(theta_2,'x');
```

The boundary conditions (3b) and (4b) are created by substituting  $x^+ = x_1^+$  ( $x=X1$ ) in the expression for  $\theta_1(x)$  and  $\theta_2(x)$  and calling it bc1. MATLAB commends are:

```
>> syms X1 x
>> bc1=subs(theta_1,x,{sym('X1')})-
subs(theta_2,x,{sym('X1')});
```

Similarly, the boundary conditions (3d) and (4c) are created by substituting  $x^+ = x_1^+$  ( $x=X1$ ) in the expression for  $\theta_1(x)$  and  $\theta_2(x)$  calling it bc2. MATLAB command is:

```
>> bc2=subs(q1,x,{sym('X1')})-subs(q2,x,{sym('X1')});
The boundary condition at the base (4d) can be created by
substituting  $x=1$  in the expressions for  $\theta_2(x)$  and call it
bc3. MATLAB command is:
```

```
>> bc3=subs(theta_2,x,1)-1;
```

The constants C1, C3, and C4 are obtained by solving the three simultaneous algebraic equations (bc1, bc2, and bc3) which written in format of solve command. MATLAB command is:

```
>> bc=solve(bc1,bc2,bc3,'c1','c3','c4');
```

The solutions for a reside in the "a-field" of bc. That is the MATLAB commands are:

```
>> c1=bc.c1;
>> c3=bc.c3;
>> c4=bc.c4;
```

To simplify the expressions for constants c1,c3, and c4 the MATLAB command simplify is used to perform symbolic simplification. The MATLAB command is:

```
>> simplify(c1);
>> simplify(c3);
>> simplify(c4);
```

The solutions for  $\theta_1(x)$  and  $\theta_2(x)$  are now recalled as theta\_1 and theta\_2 respectively. The MATLAB commands are:

```
>> theta_1=subs(theta_1,c1,{sym('c1')});
>> theta_2=subs(theta_2,c3,{sym('c3')});
>> theta_2=subs(theta_2,c4,{sym('c4')});
```

Finally the command pretty is used to prints the symbolic expression of theta\_1 and theta\_2 in a format that resembles type-set mathematics. The MATLAB command is:

```
>> pretty(theta_1);
>> pretty(theta_2);
```

Now, it is the time to find the heat flux by using the following MATLAB commands:

```
>> q2=subs(q2,c3,{sym('c3')});
q2=subs(q2,c4,{sym('c4')});
>> disp('q2'); pretty(q2); q='k*w*((T_b-T_inf)/L)*q2';
```