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Simulation and Lyapunov's Exponents Characterisation of Lorenz and Rösler Dynamics

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ABSTRACT

This study investigated the characterisation of the dynamic responses of 3-dimensional Lorenz and Rösler models by Lyapunov's exponents using popular but laborious to implement Grahm Schmidt orthogonal rules over wider range of models driven parameters. The study also verifies a new proposed model for the validation of Lyapunov's spectrum when the requisite matrix depends on positions on the model attractor. Models and the corresponding Lyapunov's spectrums were simulated by appropriately effecting Grahm Schmidt orthogonal rules and using three different detailed constant step Runge-Kutta algorithms. The FORTRAN-90 coded algorithms were validated using literature results reported by Vladimir Golovko (2003). The stability of Lyapunov's exponents estimate variation was studied in the range of estimate reset period of $2 \le \tau \le 16$ .The Lorenz model was characterised at σ =10, ρ =28, and $1 \le \beta \le 2.8$. This range covers both square and rectangular geometries. Similarly, Rösler model was characterised at $\alpha = \gamma =$ =0.2 and $2 \le \mu \le 6$. This range has potential to drive the model both periodically and chaotically depending on the choices of μ .The validation of the largest Lyapunov's exponents (λ_1) in Rösler model suffered the highest relative absolute percentage error of 14.29 while its absolute error is one of the lowest (0.01). The remaining five Lyapunov's exponents (three from Lorenz and two from Rösler) suffered relative absolute percentage error of $\le$ 2.00. Estimated Lyapunov's exponents stabilise for estimate reset period ≤ 10 .The most stable algorithms was found to be Butcher's modified fifth order followed respectively by fourth (RK4) and fifth (RK5) order. Estimation of Lyapunov's exponents' in Rösler model was found to be insensitive to algorithms due to its relative low degree of nonlinearity when compared with Lorenz model. It was established that the sum of Lyapunov's spectrum is the same as the average of trace of variation square matrix over large iteration regardless of dependence on position variable or not. This study demonstrated that the utility of Lyapunov's exponents as response characterising tool of dynamic systems driven by different parameters combination justify its laborious estimation by Grahm Schmidt method.

Keywords: *Lyapunova's Exponent, Lorenz, Rosler, Grahm Schmidt, Runge-Kutta*

1. INTRODUCTION

Simulation and characterization of nonstationary systems dynamics is an important area where research efforts need to be intensified. Nonstationary dynamical systems arise in applications, but little effort has been made in terms of the characterization of such systems, as most standard notions in nonlinear dynamics such as the Lyapunov exponents and fractal dimensions are developed for stationary dynamical systems (Ruth et al, 2008).Their paper proposed a framework for characterizing nonstationary dynamical systems. Detecting the presence of chaos in a dynamical system is an important problem that is solved by measuring the largest Lyapunov exponent (Michael et al, 1992).Lyapunov exponents quantify the exponential divergence of initially close state-space trajectories and estimate the amount of chaos in a system. Their paper presented a new method for calculating the largest Lyapunov exponent from an experimental time series. The paper reported that the largest Lyapunov exponent is an accurate method because it takes advantage of all the available data. Results of their study showed that the algorithm is fast, easy to implement, and robust to changes in the embedding dimension, size of data set, reconstruction delay, and noise level. Giovanni et al (2010) characterized the response of a chaotic system by investigating ensembles trajectories. Time-periodic stimulations were experimentally and numerically explored. Result showed that a large average response is not necessarily related to the presence of standard forms of synchronization. The stability of the response, by introducing an effective method to determine the largest nonzero eigenvalue $-\gamma_1$ of the corresponding Liouville-type operator, without the need of directly simulating it was equally studied. It was found that the exponent γ_1 is a dynamical invariant, which complements the standard characterization provided by the Lyapunov exponents. A method for characterizing the predictability of complex chaotic systems based on a generalization of the Lyapunov exponent was presented by Boffetta and Celani (1998). The method was illustrated on a toy system with two time scales and on a model of fully developed turbulence where universal features were found. The study was able to demonstrate that in systems with possess different characteristic time scales; the predictability time can be an independent quantity of the leading Lyapunov exponent. It was found that Finite Size Lyapunov Exponent is expected to converge to the leading Lyapunov exponent for very small errors while for larger errors, A(S) is decreasing with S and thus the FSLE analysis predicts an enhancement of the

predictability time at large tolerances. The authors reliably demonstrated this method using a toy model with two timescales and in a Shell Model of turbulence where universal scaling law for the error growth rate was found. According to Ying-Cheng and Zonghua (2004), a chaotic attractor from a deterministic flow must necessarily possess a neutral direction, as characterized by a null Lyapunov exponent. The authors in their paper demonstrates that for a wide class of chaotic attractors especially those having multiple scrolls in the phase space, the existence of the neutral direction can be extremely fragile in the sense that it is typically destroyed by noise of arbitrarily small amplitude. A universal scaling law quantifying the increase of the Lyapunov exponent with noise was obtained. A new three-dimensional continuous autonomous chaotic system with ten terms and three quadratic nonlinearities was presented by Qais and Mohammad (2011). The new system contains five variational parameters and exhibits Lorenz and Rossler like attractors in numerical simulations. The basic dynamical properties of the new system were analyzed by means of equilibrium points, eigenvalue structures. The authors were able to employ Lyapunov Exponent in exploring some of the basic dynamic behaviour of the system. Gleison et al paper proposed a procedure by which it is possible to synthesize Rossler and Lorenz dynamics by means of only two affine linear systems and an abrupt switching law. Comparison of different (valid) switching laws suggests that parameters of such a law behave as codimension one bifurcation parameters that can be changed to produce various dynamical regimes equivalent to those observed with the original systems. Topological analysis was employed to characterize the resulting attractors and to compare them with the original attractors. The paper provides reliable guidelines that are useful in synthesizing other chaotic dynamics using switching affine linear systems.
Extensive literature study reveals that the characterization of the dynamic responses of 3-dimensional Lorenz and R**ö**sler models by Lyapunov's exponents using a renowned but more tasking Grahm Schmidt orthogonal rules over wider range of models driven parameters has not been significantly explored. The hallmark of this paper is to address this research gap.

2. THEORY AND METHODOLOGY

The Lyapunov's spectrum of Lorenz and Rosler dynamic systems were estimated by Grahm Schmidt orthogonal rules using three different constant step Runge-Kutta methods. Marco Sandri (1996) gives details of Grahm Schmidt orthogonal rules. Similarly, the FORTRAN-90 coded algorithms were validated using literature results reported by Vladimir Golovko (2003). It is to be noted that in a chaotic system the largest Lyapunov's exponents is positive and negative or zero otherwise, see Michael et al (1993). The relevant first order system and variation rate equations are listed in equation (1) to (14).

Lorenz Weather Model:

This is a mathematical model for thermally induced fluid convection in the atmosphere proposed by Lorenz in 1993 (see Francis, 1987).

$$\dot{X} = \sigma(Y - X) \qquad (1)$$

$$\dot{Y} = \rho X - Y - XZ \qquad (2)$$

$$\dot{Z} = XY - \beta Z \qquad (3)$$

The steady solutions of the rate equations (1) to (3) were sought for numerically and simultaneously using constant step fourth, fifth and Butcher's (1964) modified fifth order Runge-Kutta methods. These three Runge-Kutta methods are tagged respectively RK4, RK5 and RK5B. In equations (1) to (3), we have X = Amplitude of fluid velocity related variable while Y and Z measures the distribution of temperature. The parameters σ and ρ are related to the Prandtl number and Rayleigh number, respectively, and the third parameter β is a geometric factor. The variation equation is given by matrix equation (4).

$$\dot{\eta} = A \bullet \eta \qquad (4)$$

In equation (4), $\dot{\eta}$ and η are variation velocity and position vectors and A is square matrix of the partial derivatives of relevant rate equations like equations (1) to (3), see equation (5). Grahm Schmidt orthogonal rules involve seeking steady solutions of equations (1) to (4) simultaneously starting from prescribed initial conditions and noting that matrix **A** may be time dependent. However it is highly recommended that the choice of initial conditions for equation (4) satisfy $|\eta(0)| = 1$. This will enable determination of length ratio given by equation (6) after a recommended time interval $t_{k+1} - t_k = \tau$ called reset period has elapsed. The localised Lyapunov's exponents (λ) can be obtained using equation (7) while necessary normalisation to unit absolute value before next time calculations can be effected using equation (8). The slope of line of best fit to plots of logarithm of absolute length ratio and time correspond to local Lyapunov's exponents (λ).

$$A = \begin{pmatrix} \frac{\delta \dot{X}}{\delta X} & \frac{\delta \dot{X}}{\delta Y} & \frac{\delta \dot{X}}{\delta Z} \\ \frac{\delta \dot{Y}}{\delta X} & \frac{\delta \dot{Y}}{\delta Y} & \frac{\delta \dot{Y}}{\delta Z} \\ \frac{\delta \dot{Z}}{\delta X} & \frac{\delta \dot{Z}}{\delta Y} & \frac{\delta \dot{Z}}{\delta Z} \end{pmatrix} \qquad (5)$$

$$\frac{d(t_{k+1})}{d(t_k)} = \frac{|\eta(\tau; t_k)|}{|\eta(O; t_k)|} \qquad (6)$$

$$d(t_{k+1}) = d(t_k).2^{\lambda t} = d(t_k).e^{\lambda t} \qquad (7)$$

$$\eta(0;t_k) = \frac{\eta(\tau;t_k)}{|\eta(\tau;t_k)|} \quad (8)$$

Substitution of equations (1) to (3) in equation (4) results in Lorenz variation equation (9).

$$\begin{Bmatrix} \dot{\eta}_1 \\ \dot{\eta}_2 \\ \dot{\eta}_3 \end{Bmatrix} = \begin{pmatrix} -\sigma & \sigma & 0 \\ \rho - Z & -1 & -X \\ Y & X & -\beta \end{pmatrix} \begin{Bmatrix} \eta_1 \\ \eta_2 \\ \eta_3 \end{Bmatrix} \quad (9)$$

It is important to note that the trace of matrix **A** in equation (9) is independent of position on the attractor of Lorenz and according to Michae (2000), it is the sum of the Lyapunov's spectrum. Thus as a useful check for the validity of Lyapunov's estimate algorithms the sum of Lyapunov's spectrum for Lorenz model must be trace (A) = $-(1+\sigma+\beta)$.

RÖSLER Chemical Reaction Model:

This is a model motivated by the dynamics of chemical reactions in a stirred tank proposed in 1976. It consists of three first order rate equations and involves three driven parameters (α, γ, μ), see equations (10) to (12).

$$\dot{X} = -(Y+Z) \quad (10)$$

$$\dot{Y} = X + \alpha Y \quad (11)$$

$$\dot{Z} = \gamma + Z(X-\mu) \quad (12)$$

Substitution of equations (10) to (12) in equation (4) results in Rösler variation equation (13).

$$\begin{Bmatrix} \dot{\eta}_1 \\ \dot{\eta}_2 \\ \dot{\eta}_3 \end{Bmatrix} = \begin{pmatrix} 0 & -1 & -1 \\ 1 & \alpha & 0 \\ Z & 0 & (X-\mu) \end{pmatrix} \begin{Bmatrix} \eta_1 \\ \eta_2 \\ \eta_3 \end{Bmatrix} \quad (13)$$

The steady solutions of the rate equations (10) to (13) were sought for numerically and simultaneously using constant step fourth, fifth and Butcher's (1964) modified fifth order Runge-Kutta methods. The calculation of the Lyapunov's exponents was done using equations (6) to (8). A useful check for the validity of Lyapunov's estimate algorithms for Rösler model proposed by the present study is given by equation (14) because the trace of matrix **A** in equation (13) is dependent on variable position (X) on the attractor. This is equivalent to average of trace of A taken over large iterate (N).

$$\text{Sum of Lyapunov's Spectrum} = \frac{1}{N}\sum_{i=1}^{N}(\alpha + X_i - \mu) \quad (14)$$

Driven Parameters and Initial Conditions Setting:
Common to all studied cases are constant time step (Δt = 0.01), transient period (1000 Δt) and steady solution period (15000 Δt). Similarly initial conditions (X, Y, Z) was set at (1, 0, 1) while initial conditions for equation (4) are (1,0,0), (0,1,0) and (0,0,1) in X, Y, and Z-directions respectively.

Lorenz Model:
Driven parameters setting are σ=10, ρ=28, and $1 \le \beta \le 2.8$. The square and rectangular geometries are covered by the range of β. However for validation of FORTRAN-90 coded algorithms $\beta = \frac{8}{3} = 2.6667$ and Lyapunov's exponent's estimation reset period (i.e. τ= **LEERP**) equal (10 Δt).

Rösler Model

Driven parameters setting are $\alpha = \gamma =$=0.2 and $2 \le \mu \le 6$. Francis (1987) recommended this range of μ to enable observations of periodic and chaotic responses. However for validation of FORTRAN-90 coded algorithms $\mu = 5.7$.

3. RESULTS AND DISCUSSION

Table 1 gives algorithms validation results referencing literature results of Vladimir Golovko (2003).

Table 1: Lyapunov's spectrum of Lorenz (β =2.6667) and Rösler (μ =5.7) models using fourth order Runge-Kutta method

	Lyapunov's spectrum and trace (A)/average trace (A) validation							
	Lorenz model				Rösler model			
	λ_1	λ_2	λ_3	Trace (A)	λ_1	λ_2	λ_3	Average trace (A)
Actual Spectrum	0.91	0.00	-14.47	-13.56	0.07	0.00	-5.39	-5.32

Current Estimate	0.92	-0.01	-14.57	-13.67	0.08	0.02	-5.42	-5.32
Absolute error	0.01	0.01	0.10	0.11	0.01	0.02	0.03	0.00
Relative absolute % error	1.10	1.00	0.69	0.81	14.29	2.00	0.56	0.00

Though the largest Lyapunov's exponents (λ_1) of Rösler model suffered the highest relative absolute percentage error of 14.29, its absolute error is one of the lowest. The remaining five Lyapunov's exponents (three from Lorenz and two from Rösler) suffered relative absolute percentage error of $\leq$ 2.00. Similarly trace (A) in Lorenz model and average trace (A) in Rösler model suffered respectively 0.81 and 0.00 relative absolute percentage error. Overall results in table 1 are acceptable as establishing the validity of algorithms developed for the present study.

The variation of estimated Lyapunov's spectrum ($\lambda_1, \lambda_2, \lambda_3$) with Lyapunov's exponent's estimation reset period (**LEERP**) for constant one full time step (**NRK1**) and constant two half-steps in one time step (**NRK2**) are given in tables 2 to 5.

Table 2:Variation of estimated Lyapunov's spectrum ($\lambda_1, \lambda_2, \lambda_3$) with Lyapunov's exponent's estimation reset period (LEERP) in Lorenz model using NRK1.

LEERP Reset Period	Runge-Kutta Algorithms								
	RK4			RK5			RK5B		
	λ_1	λ_2	λ_3	λ_1	λ_2	λ_3	λ_1	λ_2	λ_3
2	0.87	-0.01	-14.52	0.92	-0.01	-14.58	0.90	-0.01	-14.55
4	0.87	-0.01	-14.52	0.92	-0.01	-14.58	0.90	-0.01	-14.55
6	0.87	-0.01	-14.52	0.92	-0.01	-14.58	0.90	-0.01	-14.55
8	0.87	-0.01	-14.52	0.92	-0.01	-14.58	0.90	-0.01	-14.55
10	0.87	-0.01	-14.52	0.92	-0.01	-14.58	0.89	-0.01	-14.55
12	0.86	-0.01	-14.52	0.92	-0.01	-14.58	0.89	-0.01	-14.55
14	0.87	-0.01	-14.52	0.92	-0.01	-14.58	0.89	-0.01	-14.55
16	0.86	-0.01	-14.52	0.92	-0.01	-14.58	0.89	-0.01	-14.55

Though referencing table 2 , there is observed variation in Lyapunov's spectrum over Runge-Kutta algorithms, the trace (A) remain the same (-13.67) for algorithms and range of Lyapunov's exponent's estimation reset period (**LEERP**).

Table 3: Variation of estimated Lyapunov's spectrum ($\lambda_1, \lambda_2, \lambda_3$) with Lyapunov's exponent's estimation reset period (LEERP) in Lorenz model using NRK2.

LEERP Reset Period	Runge-Kutta Algorithms								
	RK4			RK5			RK5B		
	λ_1	λ_2	λ_3	λ_1	λ_2	λ_3	λ_1	λ_2	λ_3
2	0.92	-0.01	-14.57	0.89	0.00	-14.55	0.90	0.00	-14.56
4	0.92	-0.01	-14.57	0.89	0.00	-14.55	0.90	0.00	-14.56
6	0.92	-0.01	-14.57	0.89	0.00	-14.55	0.90	0.00	-14.56
8	0.92	-0.01	-14.57	0.89	0.00	-14.55	0.90	0.00	-14.56
10	0.92	-0.01	-14.57	0.89	0.00	-14.55	0.90	0.00	-14.56
12	0.92	-0.01	-14.58	0.89	0.00	-14.55	0.90	0.00	-14.56
14	0.92	-0.01	-14.57	0.88	0.00	-14.55	0.90	0.00	-14.56
16	0.92	-0.01	-14.57	0.89	0.00	-14.55	0.90	0.00	-14.56

Table 3 refers there is observed variation in Lyapunov's spectrum over Runge-Kutta algorithms however the trace (A) remain the same (-13.67) for algorithms and range of Lyapunov's exponent's estimation reset period (**LEERP**).

Table 4: Variation of estimated Lyapunov's spectrum ($\lambda_1, \lambda_2, \lambda_3$) with Lyapunov's exponent's estimation reset period (LEERP) in Rösler model using NRK1.

LEERP Reset Period	Runge-Kutta Algorithms								
	RK4			RK5			RK5B		
	λ_1	λ_2	λ_3	λ_1	λ_2	λ_3	λ_1	λ_2	λ_3
2	0.08	0.02	-5.42	0.08	0.02	-5.42	0.08	0.02	-5.42
4	0.08	0.02	-5.42	0.08	0.02	-5.42	0.08	0.02	-5.42
6	0.08	0.02	-5.42	0.08	0.02	-5.42	0.08	0.02	-5.42

8	0.08	0.02	-5.42	0.08	0.02	-5.42	0.08	0.02	-5.42
10	0.08	0.02	-5.42	0.08	0.02	-5.42	0.08	0.02	-5.42
12	0.08	0.02	-5.42	0.08	0.02	-5.42	0.08	0.02	-5.42
14	0.08	0.01	-5.42	0.08	0.01	-5.42	0.08	0.01	-5.42
16	0.08	0.01	-5.42	0.08	0.01	-5.42	0.08	0.01	-5.42

Table 4 refers, there is **no** observed variation in Lyapunov's spectrum over Runge-Kutta algorithms, the trace (A) remain the same (-5.32) for algorithms and range of Lyapunov's exponent's estimation reset period (**LEERP**).

Table 5: Variation of estimated Lyapunov's spectrum ($\lambda_1, \lambda_2, \lambda_3$) with Lyapunov's exponent's estimation reset period (LEERP) in Rösler model using NRK2.

LEERP Reset Period	Runge-Kutta Algorithms								
	RK4			RK5			RK5B		
	λ_1	λ_2	λ_3	λ_1	λ_2	λ_3	λ_1	λ_2	λ_3
2	0.08	0.01	-5.45	0.08	0.01	-5.45	0.08	0.01	-5.45
4	0.08	0.01	-5.45	0.08	0.01	-5.45	0.08	0.01	-5.45
6	0.08	0.01	-5.45	0.08	0.01	-5.45	0.08	0.01	-5.45
8	0.08	0.01	-5.45	0.08	0.01	-5.45	0.08	0.01	-5.45
10	0.08	0.01	-5.45	0.08	0.01	-5.45	0.08	0.01	-5.45
12	0.07	0.02	-5.45	0.07	0.02	-5.45	0.07	0.02	-5.45
14	0.08	0.02	-5.45	0.08	0.02	-5.45	0.08	0.02	-5.45
16	0.07	0.02	-5.45	0.07	0.02	-5.45	0.07	0.02	-5.45

Table 5 refers there is observed insignificant variation in Lyapunov's spectrum over Runge-Kutta algorithms however the trace (A) remain the same (-5.36) for algorithms and range of Lyapunov's exponent's estimation reset period (**LEERP**).The variation of estimated largest Lyapunov's exponents (**ELLE**= λ_1) with Lyapunov's exponent's estimation reset period (**LEERP**) for constant one full time step (**NRK1**) and constant two half-steps time step (**NRK2**) are given in figures 1 to 2. Similarly, the variation of estimated largest Lyapunov's exponents (**ELLE**= λ_1) with parameters are given in figures 3 to 6.

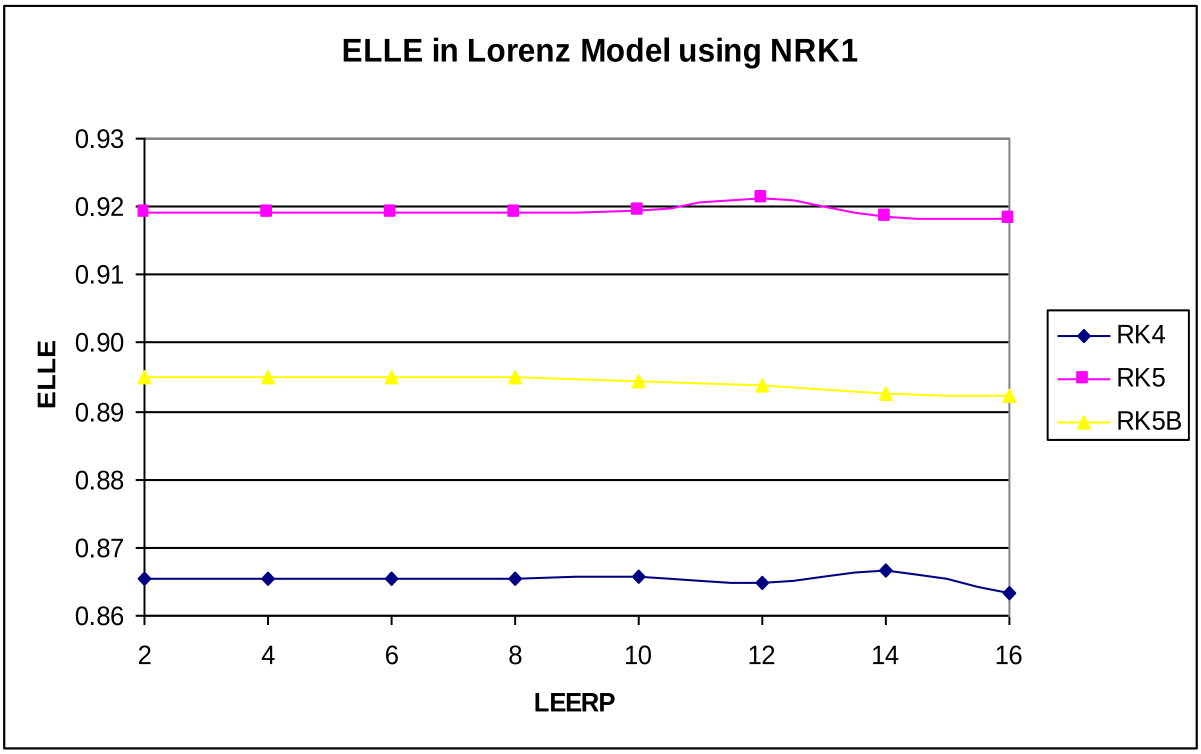

Figure 1: Variation of estimated largest Lyapunov's exponents (ELLE= λ_1) with Lyapunov's exponent's estimation reset period (LEERP) in Lorenz model using NRK1.

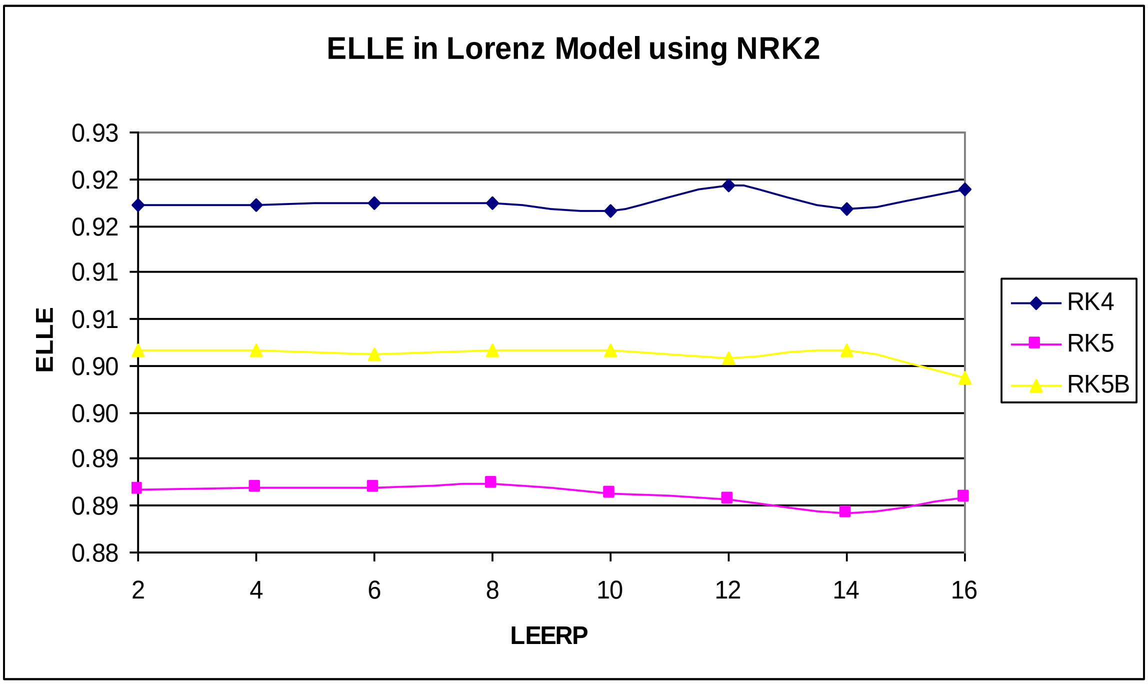

Figure 2: Variation of estimated largest Lyapunov's exponents (ELLE= λ_1) with Lyapunov's exponent's estimation reset period (LEERP) in Lorenz model using NRK2.

Figures 1 and 2 refers, the estimated largest Lyapumov's exponents (**ELLE**= λ_1) remain fairly stable up to LEERP of ten (10). The ELLE by RK4 are all lower than ELLE by RK5B while ELLE by RK5 are all higher over the range of LEERP in figure 1.However role play change observed between RK4 and RK5 in figure 2 compared with figure 1 in term of position relative to ELLE variation by RK5B.

The estimated largest Lyapumov's exponents (**ELLE**= λ_1) in Rösler model remain constant (0.0.08) over all LEERP algorithms and using NRK1. However at NRK2 ELLE of 0.07 was recorded at LEERP values of twelve (12) and sixteen (16) across algorithms. It is concluded that estimation of Lyapunov's spectrum in Rösler model is less sensitive to algorithms.

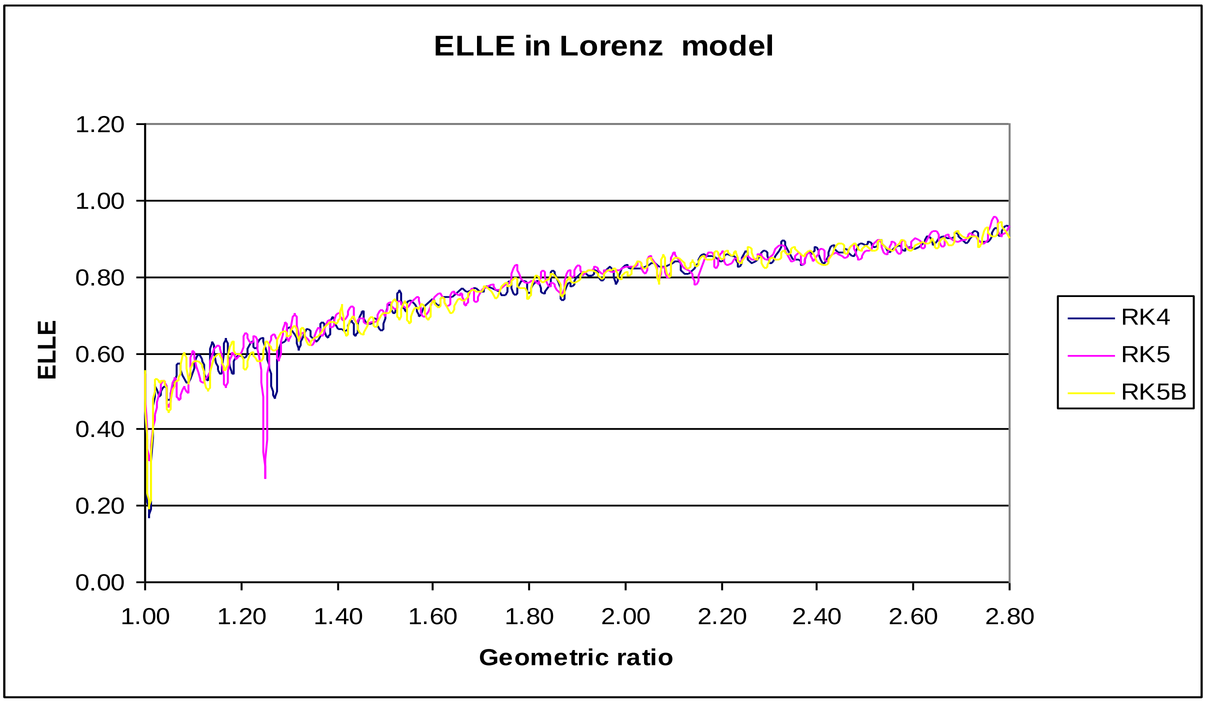

Figure 3: Variation of ELLE with geometric ratio (β) in Lorenz model using NRK1 and LEERP of ten (10).

Figure 3 refers, it is concluded that all the geometric ratio ($1 \leq \beta \leq 2.8$) can drive the Lorenz model chaotically because ELLE are positive. The zoomed of variation of ELLE with geometric ratio in Lorenz for $1.1 \leq \beta \leq 1.4$ using NRK1 and LEERP of ten is given in figure 4. This is done to enable the magnification of variation of ELLE with algorithms for the β-region that suffered significant variation in figure 3. Figure 4 refers, the most stable algorithms is RK5B followed by RK4 and RK5 respectively.

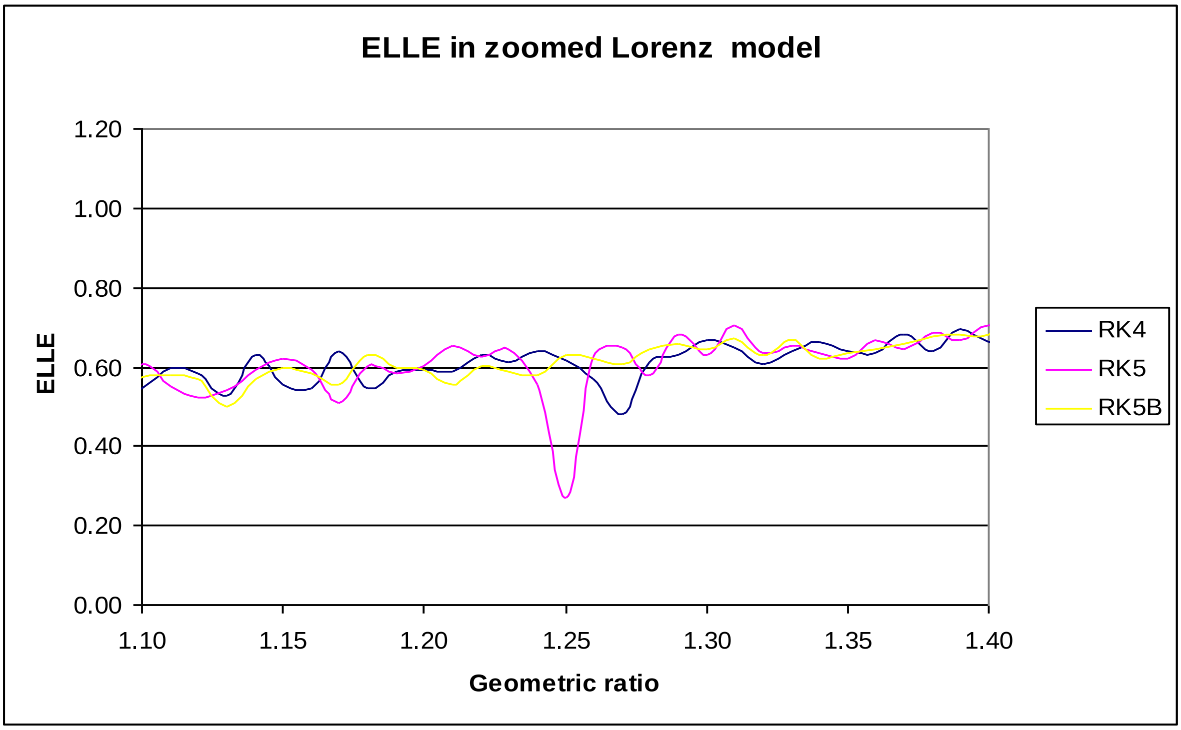

Figure 4: Variation of ELLE with geometric ratio (β) in Lorenz model using NRK1 and LEERP of ten (10) in the region $1.1 \leq \beta \leq 1.4$.

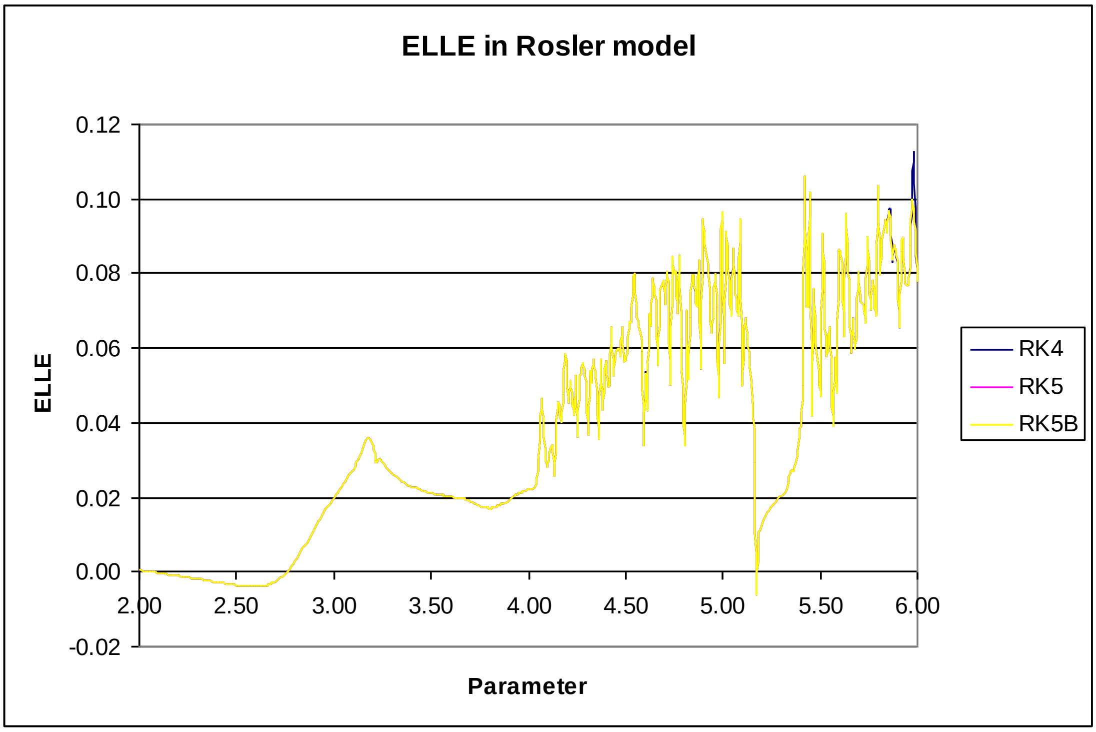

Figure 5: Variation of ELLE with parameter (μ) in Rösler model using NRK1 and LEERP of ten (10).

Figure 5 refers, it is concluded that there are possibilities of period and chaotic responses in the parameter range ($2 \leq \mu \leq 6$) because ELLE takes both negative and positive values. The three algorithms (RK4, RK5 and RK5B) are fairly equally stable over the studied parameter range. The zoomed of variation of ELLE with parameter in Rösler model for $5.8 \leq \mu \leq 6$ using NRK1 and

LEERP of ten is given in figure 6. The magnification of variation of ELLE with algorithms for the selected μ -region shows that the variation is less significant with respect to algorithms. However, algorithms (RK4) becomes unstable for higher parameter (μ) compared with RK5 and RK5B.

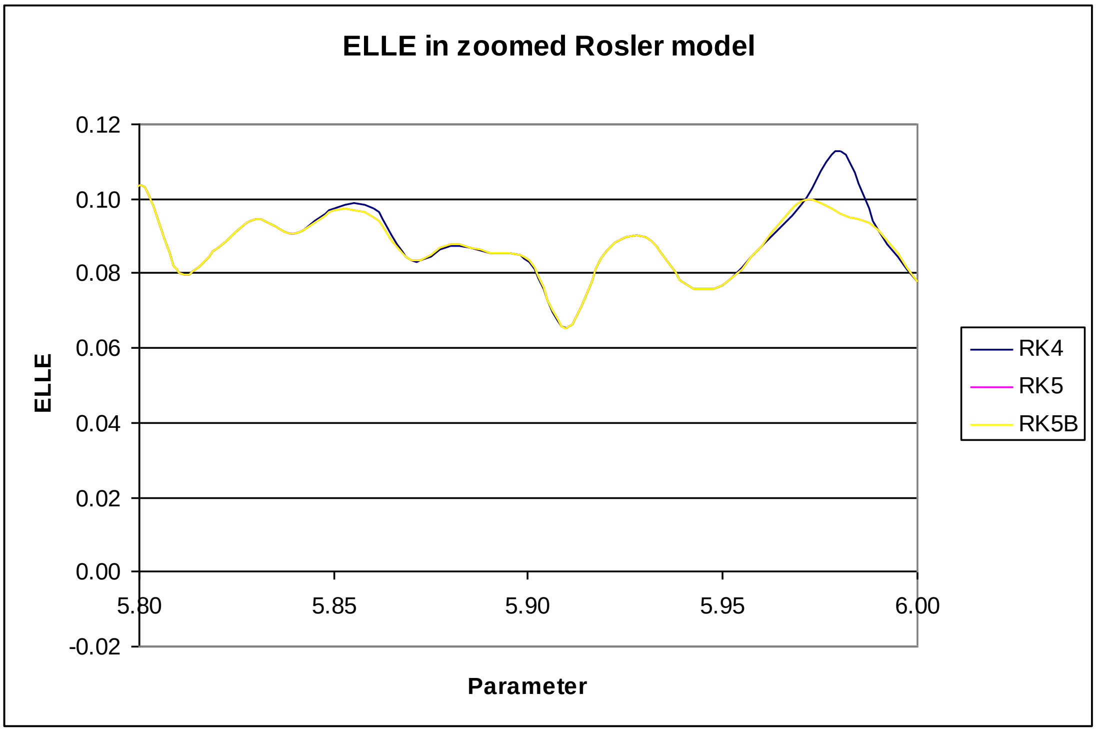

Figure 6: Variation of ELLE with parameter (μ) in Rösler model using NRK1 and LEERP of ten (10) in the region $5.8 \leq \mu \leq 6$.

4. CONCLUSIONS

This study shows that the utility of Lyapunov's exponents as response characterising tool of dynamic systems driven by different parameters combination justify its laborious estimation by Grahm Schmidt method. Also, this study shows that accuracy of estimated Lyapunov's exponents depends on dynamic system degree of nonlinearity, algorithms, estimation reset period and time stepping details. Comparatively with fourth and fifth order Runge-Kutta algorithms, the coding of modified Butcher's fifth order algorithms is laborious. However it is the most stable to use to estimate Lyapunov's exponents in all the cases studied. Furthermore, this study established that the sum of Lyapunov's spectrum is the same as the average of trace of variation square matrix over large iteration regardless of dependence on position variable or not.

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