Journal of Engineering & Technology
CENTRE OF PROFESSIONAL RESEARCH PUBLICATIONS
ISSN 2049-3444

International Journal of Engineering and Technology Volume 4 No. 8, August, 2014

Simulation of Oscillators Dynamics using Selected Versions of Fourth Order Runge-Kutta Scheme

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ABSTRACT

This study investigated the simulation performance from zero initial conditions across the transient and steady states of six different versions of fourth order Runge-Kutta schemes (RK41, RK42, RK43, RK44, RK45 & RK46-stable and unstable) on cases of linear and nonlinear dynamics. Validation cases were obtained from [1-3] including the periodically $(q, g \equiv 2,1.47)$ and chaotically $(q, g \equiv 4,1.5)$ responding nonlinear excited pendulum at drive frequency of $(\omega_D = 2/3)$. The simulated results of linear dynamics correlate quantitatively well for all the schemes when compared with the results reported by [1] except for RK44 which was reported less stable by [3]. However for a reduced simulation time step of 0.0025 the simulated results by RK44 improved drastically towards the exact results as well as the results prescribed by other schemes. The schemes performance replicate for the case of periodically behaving nonlinear pendulum. However the simulated results of the chaotic counterpart of the nonlinear pendulum lacked correlation, but all the schemes produced qualitatively the same Poincare section that is qualitatively the same as reported by [2]. Provided the scheme stability is guaranteed, the study results can be adopted in the characterization of dynamic systems.

Keywords: *RK4 Versions, Oscillator Dynamics, Chaotic, Simulation time step and Poincare Section*

1. INTRODUCTION

The concept of oscillation system is a key universal phenomenon in diverse fields of study. In the online article posted by [4], oscillation can be described as a periodic fluctuation or back and forth movement between two objects. The device that experiences this back and forth movement is generally termed as oscillator. The importance of oscillators' dynamics in Theoretical physics, Electrical/Electronic Engineering and Mechanical Engineering just to mention a few cannot be overemphasized. In theoretical physics, interesting efforts have been made by researchers in the field to characterize oscillator dynamics and their applications. The dynamics of coupled van der pol and Duffing oscillators as a typical coupled system with various attractors has been examined [5]. The results obtained showed that coupled system is rich in dynamic phases for various values of the system parameters. The author's paper finally submitted that chaotic region posses islands of periodic windows exhibiting period-doubling characteristics. The dynamic and synchronization characteristic of two unidirectional coupled double-well Duffing oscillators has been critically investigated [6]. Numerical simulations were carried out using fourth order Runge-Kutta and the results obtained imply evidence of transition to synchronization through a bistable state in which a torus co-exists with a cross-well chaotic attractor. Deductions made from this coexistence of periodic and chaotic attractor dynamic behaviour will be of immense applications for transport phenomenon in inertial ratchets. [7] Studied the nonlinear dynamics of plasma oscillations that is modelled by a forced modified Van der Pol Duffing oscillator. The bifurcation sequences obtained by the model were successfully performed through numerical method using fourth order Runge-Kutta. The outcome of the author's study is of great benefit for control of high amplitude oscillations which are often the cause of instability in plasma physics. In the same vein, there has been a very robust research on the applicability of oscillator dynamics in the field of Electrical and Electronic engineering especially in the area of control. [8] Considered the chaos-based dynamics of a weak signal detection method with Duffing oscillator. The authors established that chaotic detecting approach is only useful for single periodic signal detection with very short frequency band whereas extremely difficult when the frequency band is large. In view of this shortcoming, the authors then proposed Empirical Mode Decomposition (EMD) method for analysis of the system dynamics. Studies from the literature informed that complex signals are often used in the digital processing of communication and radar systems. Such systems are prone to nonlinear dynamical behaviours. [9] Developed a new complex Duffing oscillator that can be employed in detection of complex signals using Monte-Carlo simulation technique. The study has shown that the developed system is capable of effectively detecting complex single frequency signals and linear frequency modulation signals. [10-14] papers have also demonstrated the usefulness of Oscillator's system dynamics in the field of electrical and electronic technology. Furthermore, oscillator dynamics play prominent applications in mechanical engineering discipline. Frequency responses, bifurcation and chaos dynamics of a SDOF oscillator with different parameters of Bouc-Wen model have been studied [15]. In order to numerically simulate the hysteresis, the classical Runge-Kutta method is used. It has been shown in the study that domains where chaotic behaviour of the oscillator with hysteresis is possible can be found in planes defined by amplitude of external excitation and the hysteretic parameter.

The wide dynamical applications of oscillator systems in mechanical engineering have also been reinforced in [16-20] papers. In nonlinear dynamics, the utility of Runge-Kutta numerical simulation is a widely accepted technique. Runge-Kutta methods are important family of implicit and explicit iterative methods [21]. They are generally used for temporal discretizations for the approximation of solutions of ordinary differential equations. Fourth order Runge-Kutta method has been found to be one of the most satisfactory Runge-Kutta methods for oscillator dynamics characterization. In [16], the dynamics of a coupled nonlinear system consisting of a Van der Pol oscillator coupled to a driven Duffing resonator have been examined using fourth order Runge-Kutta. Findings revealed that the system dynamic is highly chaotic and strongly depend on parameters variation. Similarly, numerical simulations of nonlinear oscillator systems have been performed using fourth order Runge-Kutta method with a constant step. Research results obtained by [17] showed that the dynamic parameter space diagram of the non-ideal Duffing oscillator is characterized by a vast quantity of periodic structure produced by Arnold tongues. [6-7] and many more literature works have equally demonstrated the general richness of results produced in oscillator dynamics characterization using fourth order Runge-Kutta as numerical simulation tool.

Although [3] utilized the fourth order Runge-Kutta versions in Lorenz system, significant efforts have not been made in the literature on the choice of fourth order Runge-Kutta versions in numerical simulation of nonlinear oscillator dynamics. In view of this research gap, the present paper considers six different versions of fourth order Runge-Kutta schemes (RK41, RK42, RK43, RK44, RK45 & RK46-stable and unstable) for characterizing nonlinear oscillator dynamics.

2. METHODOLOGY

The present study utilised two existing oscillators (linear and non-linear). The linear oscillator is adopted from [1] while its nonlinear counterpart is adopted from [2]. Equation (1) described the linear oscillator under a step and ram excitations-F (t) and equation (2) gives the description of the nonlinear non-dimensional and one dimensional governing equation of the damped, sinusoidally driven pendulum. It is important to note that q is the damping quality parameter, g is the forcing amplitude, ω_D is the drive frequency and t represent time.

$$4\ddot{x} + 2000x = F(t) \tag{1}$$

Where:

$$F(t) = 100.0;\ 0.0 \le t \le 0.10,$$
$$F(t) = -1000.0t + 200.0;\ 0.10 \le t \le 0.20$$
$$F(t) = 0.0;\ t \ge 0.20$$

and

$$\frac{d^2\theta}{dt} + \frac{1}{q}\frac{d\theta}{dt} + \sin(\theta) = g\cos(\omega_D t) \tag{2}$$

Simulation of equations (1) and (2) with Runge-Kutta scheme demands first order pair transformation under the assumptions ($x_1 = linear\ displacement$, $x_2 = linear\ velocity$, $\theta_1 = angular\ displacement$ and $\theta_2 = angular\ velocity$). Under these transformations equations (3) & (4) and (5) & (6) are obtained correspondingly for the linear and non-linear oscillator.

$$\dot{x}_1 = x_2 \tag{3}$$

$$\dot{x}_2 = 0.25F(t) - 500x_1 \tag{4}$$

$$\dot{\theta}_1 = \theta_2 \tag{5}$$

$$\dot{\theta}_2 = g\cos(\omega_D t) - \frac{1}{q}\theta_2 - \sin(\theta_1) \tag{6}$$

According to [22], the numerical solution of equations (1) and (2) can be sought using equation (7), with ϕ being an incremental weighting function. The general form for ϕ is given by equation (8) in which the slope estimate of ϕ is used to extrapolate from an old value y_i to a new value y_{i+1} over a step size h.

$$y_{i+1} = y_i + \phi h \tag{7}$$

$$\phi = c_1 K_1 + c_2 K_2 + \cdots + c_n K_n \tag{8}$$

The functions K_1 to K_4 for the fourth order Runge-Kutta scheme are given by equations (9) to (12). The corresponding predicting formula is given by equation (13) with time step- h .

$$K_1 = f(x_i, y_i) \tag{9}$$

$$K_2 = f(x_i + a_2 h, y_i + b_{21} K_1) \tag{10}$$

$$K_3 = f(x_i + a_3 h, y_i + b_{31} K_1 + b_{32} K_2) \tag{11}$$

$$K_4 = f(x_i + a_4 h, y_i + b_{41} K_1 + b_{42} K_2 + b_{43} K_3) \tag{12}$$

$$y_{i+1} = y_i + h\{c_1 K_1 + c_2 K_2 + c_3 K_3 + c_4 K_4\} \tag{13}$$

The coefficients details as given in [3] that investigated stability of Runge-Kutta methods are provided in tables 1 and 2 for the specific cases of fourth order Runge-Kutta scheme.

Table 1: Butcher's tableu for general fourth order scheme

$$\begin{array}{c|cccc} 0 & 0 & 0 & 0 & 0 \\ a_2 & b_{21} & 0 & 0 & 0 \\ a_3 & b_{31} & b_{32} & 0 & 0 \\ a_4 & b_{41} & b_{42} & b_{43} & 0 \\ \hline & c_1 & c_2 & c_3 & c_4 \end{array}$$

Table 2: Coefficients for the selected six cases of Fourth order Runge-Kutta scheme.

Coefficient	Selected Fourth order Runge-Kutta Scheme					
	RK41	RK42	RK43	RK44	RK45	RK46
a_2	0.50000	0.33333	0.25000	-0.50000	0.50000	1.00000
a_3	0.50000	0.66667	0.50000	0.50000	0.00000	0.50000
a_4	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000
b_{21}	0.50000	0.33333	0.25000	-0.50000	0.50000	1.00000
b_{31}	0.00000	-0.33333	0.00000	0.75000	-1.00000	0.37500
b_{32}	0.50000	1.00000	0.50000	-0.25000	1.00000	0.12500
b_{41}	0.00000	1.00000	1.00000	-2.00000	-1.00000	-0.50000
b_{42}	0.00000	-1.00000	-2.00000	1.00000	1.50000	-0.50000
b_{43}	1.00000	1.00000	2.00000	-2.00000	0.50000	2.00000
c_1	0.16667	0.12500	0.16667	0.16667	0.08333	0.16667
c_2	0.33333	0.37500	0.00000	0.00000	0.66667	0.00000
c_3	0.33333	0.37500	0.66667	0.66667	0.08333	0.66667
c_4	0.16667	0.12500	0.16667	0.16667	0.16667	0.16667

2.1 Simulation Parameters

The two oscillators were simulated from the same initial conditions (0, 0). The linear oscillator total simulation period is 0.48-unit at constant time step of 0.02-unit to enable comparison of simulated results with the exact results reported by [1]. However the nonlinear oscillator was simulated for one case each of parameters (q, g) combination leading to periodic $(q, g \equiv 2, 1.47)$ and chaotic $(q, g \equiv 4, 1.5)$ responses as reported by [2] for fixed drive frequency $\omega_D = 2/3$, constant simulation time step $(h = T_D/500)$ where $T_D = 2\pi/\omega_D$. The simulation was executed for 2010-excitation periods comprising 10-periods of transient and 2000-peiods of steady solutions respectively.

3. RESULTS AND DISCUSSION

Table 3 refers. The displacement component of the simulated results for the linear oscillator dynamics by all the fourth order Runge-Kutta schemes compared quantitatively well with the exact results reported by [1] with the exception of the simulated results by RK44. However this observation agreed very well with the poor stability of this scheme (RK44) as reported by [3] when compared with its counterpart fourth order scheme. Similarly, table 4 provided the corresponding simulated velocity components of the linear oscillator dynamics that are quantitatively the same while figure 1 summarised the associated phase plots. Furthermore, tables 5, 6 and figure 2 refer the simulated displacement and velocity results by RK44 gravitate toward the corresponding exact result with decreasing simulation time step for all simulation time. This is suggestive that RK44 can simulate the dynamics of the linear oscillator accurately as any of its counterparts (RK41, RK42, RK43, RK5 and RK46) but at the expense of higher computation resources. The simulation time step must be relatively smaller and appropriate. Therefore, except it is solely necessary and affordable, it is not advisable to simulate system dynamics with RK44 because its use is prone to gross computational instability errors.

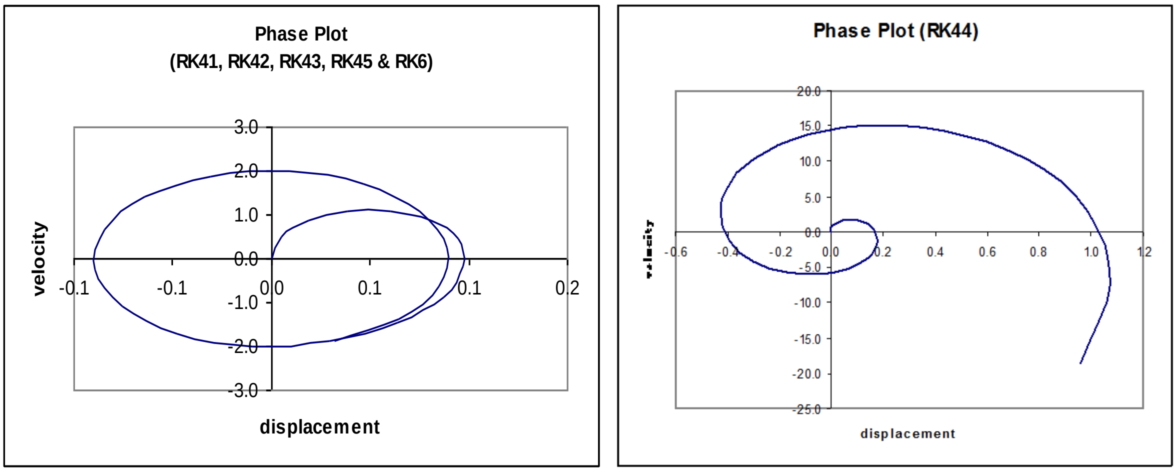
Table 3: Comparison of the Exact and simulated results of linear oscillator dynamics (displacement component only)

Simulation Time	Exact Result	Simulated Results by Fourth Order Schemes					
		RK41	RK42	RK43	RK44	RK45	RK46
0.00	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000
0.02	0.00492	0.00492	0.00492	0.00492	-0.00158	0.00492	0.00492
0.04	0.01870	0.01869	0.01869	0.01869	0.00763	0.01869	0.01869
0.06	0.03864	0.03862	0.03862	0.03862	0.02850	0.03862	0.03862
0.08	0.06082	0.06079	0.06079	0.06079	0.05978	0.06079	0.06079
0.10	0.08086	0.08083	0.08083	0.08083	0.09766	0.08083	0.08083
0.12	0.09451	0.09447	0.09447	0.09447	0.13616	0.09447	0.09447
0.14	0.09743	0.09741	0.09741	0.09741	0.16616	0.09741	0.09741
0.16	0.08710	0.08709	0.08709	0.08709	0.17667	0.08709	0.08709
0.18	0.06356	0.06359	0.06359	0.06359	0.15790	0.06359	0.06359
0.20	0.02949	0.02956	0.02956	0.02955	0.10357	0.02955	0.02955
0.22	-0.01005	-0.00995	-0.00995	-0.00996	0.01299	-0.00995	-0.00996
0.24	-0.04761	-0.04750	-0.04750	-0.04751	-0.10556	-0.04750	-0.04751
0.26	-0.07581	-0.07571	-0.07571	-0.07571	-0.23442	-0.07571	-0.07571
0.28	-0.08910	-0.08903	-0.08903	-0.08903	-0.34878	-0.08903	-0.08903
0.30	-0.08486	-0.08485	-0.08485	-0.08485	-0.42009	-0.08485	-0.08485
0.32	-0.06393	-0.06400	-0.06400	-0.06399	-0.42108	-0.06400	-0.06399
0.34	-0.03043	-0.03056	-0.03056	-0.03056	-0.33200	-0.03056	-0.03056
0.36	0.00906	0.00887	0.00887	0.00888	-0.14693	0.00887	0.00888
0.38	0.04677	0.04656	0.04656	0.04657	0.12110	0.04656	0.04657
0.40	0.07528	0.07510	0.07510	0.07510	0.43763	0.07509	0.07510
0.42	0.08898	0.08886	0.08886	0.08887	0.74822	0.08886	0.08887
0.44	0.08518	0.08516	0.08516	0.08516	0.98459	0.08516	0.08516
0.46	0.06463	0.06473	0.06473	0.06472	1.07538	0.06472	0.06472
0.48	0.03136	0.03157	0.03157	0.03156	0.96067	0.03156	0.03156

Table 4: Simulated results of linear oscillator dynamics (velocity component only).

Simulation Time	Simulated Results by Fourth Order Schemes					
	RK41	RK42	RK43	RK44	RK45	RK46
0.00	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000
0.02	0.48333	0.48333	0.48334	0.52501	0.48333	0.48334
0.04	0.87161	0.87161	0.87162	1.07467	0.87161	0.87162
0.06	1.08852	1.08852	1.08853	1.54649	1.08852	1.08853
0.08	1.09146	1.09146	1.09146	1.81762	1.09145	1.09146
0.10	0.87985	0.87985	0.87984	1.77410	0.87985	0.87984
0.12	0.44616	0.44616	0.44614	1.28783	0.44616	0.44614
0.14	-0.17346	-0.17346	-0.17349	0.28407	-0.17346	-0.17349
0.16	-0.85719	-0.85719	-0.85723	-1.16474	-0.85719	-0.85723
0.18	-1.47065	-1.47065	-1.47069	-2.87145	-1.47064	-1.47069
0.20	-1.89324	-1.89324	-1.89327	-4.54151	-1.89323	-1.89327
0.22	-1.99278	-1.99278	-1.99278	-5.75640	-1.99276	-1.99278
0.24	-1.70060	-1.70060	-1.70057	-6.07295	-1.70058	-1.70057
0.26	-1.07419	-1.07419	-1.07412	-5.17443	-1.07417	-1.07412

0.28	-0.23671	-0.23670	-0.23661	-2.91598	-0.23670	-0.23661
0.30	0.64721	0.64722	0.64732	0.59576	0.64722	0.64732
0.32	1.40382	1.40382	1.40391	4.95559	1.40381	1.40391
0.34	1.88442	1.88442	1.88448	9.46372	1.88440	1.88448
0.36	1.99456	1.99456	1.99457	13.19409	1.99453	1.99457
0.38	1.71265	1.71264	1.71259	15.13017	1.71261	1.71259
0.40	1.09413	1.09412	1.09402	14.35792	1.09410	1.09402
0.42	0.26062	0.26061	0.26046	10.29037	0.26060	0.26046
0.44	-0.62403	-0.62404	-0.62420	2.88453	-0.62404	-0.62420
0.46	-1.38592	-1.38593	-1.38607	-7.19831	-1.38591	-1.38607
0.48	-1.87532	-1.87532	-1.87542	-18.53867	-1.87529	-1.87542



(a) (b)

Figure 1: Phase plots of the linear oscillator dynamics with different selected Runge-Kutta scheme.

Table 5: Comparison of the Exact and Simulated results (by RK44) of linear oscillator dynamics for different simulation steps (displacement component only).

Simulation Time	Exact Result	Simulated Results by RK44 for different simulation steps				
		$h = 0.02$	$h = 0.01$	$h = 0.005$	$h = 0.004$	$h = 0.0025$
0.00	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000
0.02	0.00492	-0.00158	0.00173	0.00336	0.00368	0.00415
0.04	0.01870	0.00763	0.01374	0.01641	0.01690	0.01761
0.06	0.03864	0.02850	0.03496	0.03716	0.03752	0.03799
0.08	0.06082	0.05978	0.06227	0.06195	0.06178	0.06147
0.10	0.08086	0.09766	0.09083	0.08600	0.08499	0.08344
0.12	0.09451	0.13616	0.11488	0.10424	0.10221	0.09923
0.14	0.09743	0.16616	0.12741	0.11105	0.10810	0.10389
0.16	0.08710	0.17667	0.12215	0.10223	0.09884	0.09411
0.18	0.06356	0.15790	0.09563	0.07648	0.07346	0.06935
0.20	0.02949	0.10357	0.04831	0.03583	0.03417	0.03207
0.22	-0.01005	0.01299	-0.01492	-0.01435	-0.01368	-0.01246
0.24	-0.04761	-0.10556	-0.08387	-0.06507	-0.06138	-0.05601

0.26	-0.07581	-0.23442	-0.14498	-0.10606	-0.09930	-0.08984
0.28	-0.08910	-0.34878	-0.18446	-0.12835	-0.11927	-0.10685
0.30	-0.08486	-0.42009	-0.19109	-0.12614	-0.11628	-0.10309
0.32	-0.06393	-0.42108	-0.15892	-0.09820	-0.08967	-0.07860
0.34	-0.03043	-0.33200	-0.08923	-0.04842	-0.04353	-0.03756
0.36	0.00906	-0.14693	0.00871	0.01458	0.01391	0.01236
0.38	0.04677	0.12110	0.11848	0.07895	0.07168	0.06151
0.40	0.07528	0.43763	0.21891	0.13172	0.11814	0.10003
0.42	0.08898	0.74822	0.28782	0.16139	0.14332	0.11986
0.44	0.08518	0.98459	0.30648	0.16036	0.14097	0.11640
0.46	0.06463	1.07538	0.26404	0.12674	0.11006	0.08956
0.48	0.03136	0.96067	0.16079	0.06511	0.05527	0.04392

Table 6: Comparison of the simulated results by RK42 and RK44 of linear oscillator dynamics for different simulation steps (velocity component only).

Simulation Time	RK42 Result	Simulated Results by RK44 for different simulation steps				
		$h = 0.02$	$h = 0.01$	$h = 0.005$	$h = 0.004$	$h = 0.0025$
0.00	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000
0.02	0.48333	0.52501	0.50937	0.49800	0.49536	0.49116
0.04	0.87161	1.07467	0.98267	0.92899	0.91779	0.90076
0.06	1.08852	1.54649	1.31759	1.20112	1.17819	1.14419
0.08	1.09146	1.81762	1.42525	1.24830	1.21521	1.16717
0.10	0.87985	1.77410	1.25160	1.04554	1.00918	0.95766
0.12	0.44616	1.28783	0.74301	0.56677	0.53848	0.49997
0.14	-0.17346	0.28407	-0.09433	-0.16103	-0.16725	-0.17302
0.16	-0.85719	-1.16474	-1.13204	-1.00731	-0.97851	-0.93412
0.18	-1.47065	-2.87145	-2.18233	-1.80857	-1.73746	-1.63417
0.20	-1.89324	-4.54151	-3.02858	-2.39921	-2.28819	-2.13120
0.22	-1.99278	-5.75640	-3.41589	-2.59494	-2.45835	-2.26932
0.24	-1.70060	-6.07295	-3.14695	-2.27780	-2.14205	-1.95856
0.26	-1.07419	-5.17443	-2.18635	-1.47698	-1.37651	-1.24582
0.28	-0.23671	-2.91598	-0.63744	-0.32172	-0.29160	-0.25997
0.30	0.64721	0.59576	1.25969	0.97797	0.91133	0.81121
0.32	1.40382	4.95559	3.15916	2.16839	1.99613	1.75652
0.34	1.88442	9.46372	4.66641	3.00062	2.73759	2.38275
0.36	1.99456	13.19409	5.41347	3.28135	2.96745	2.55394
0.38	1.71265	15.13017	5.13917	2.91582	2.61143	2.22048
0.40	1.09413	14.35792	3.75829	1.93389	1.70936	1.43193
0.42	0.26062	10.29037	1.40379	0.49245	0.41334	0.33050
0.44	-0.62403	2.88453	-1.57045	-1.14765	-1.03677	-0.87427
0.46	-1.38592	-7.19831	-4.63242	-2.66741	-2.35694	-1.94508
0.48	-1.87532	-18.53867	-7.15779	-3.75035	-3.27386	-2.66338

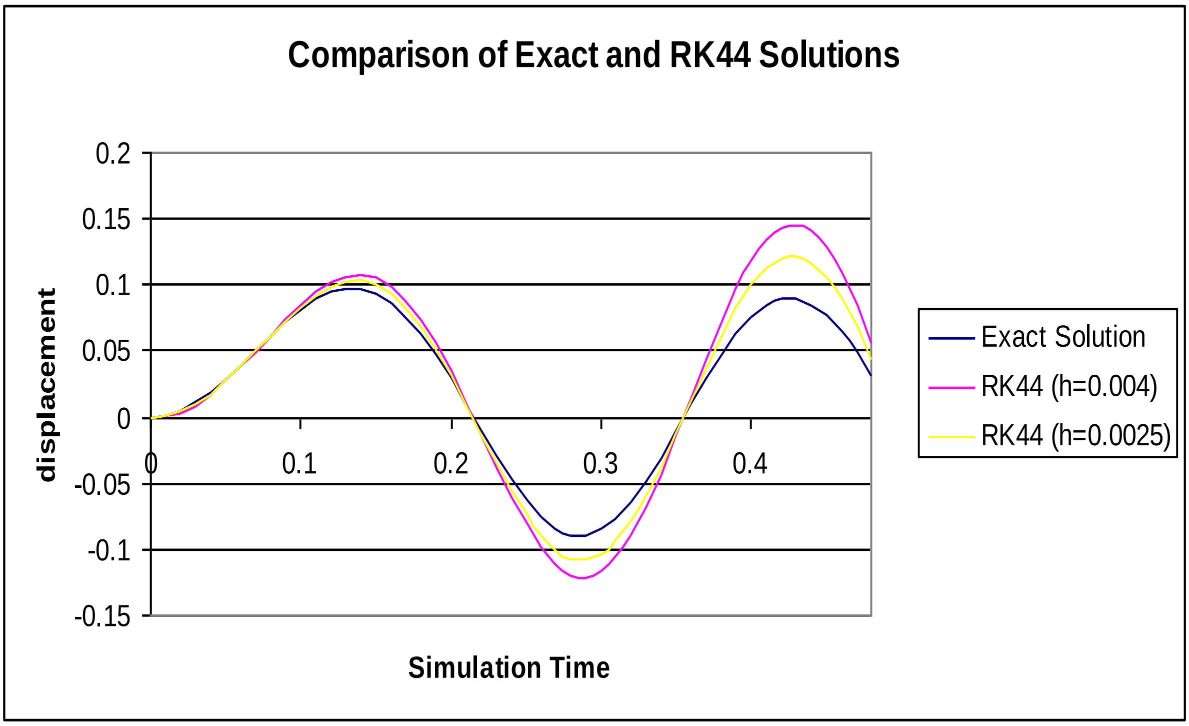
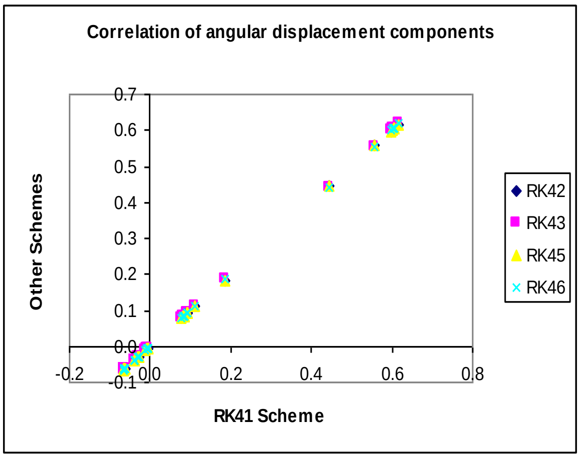
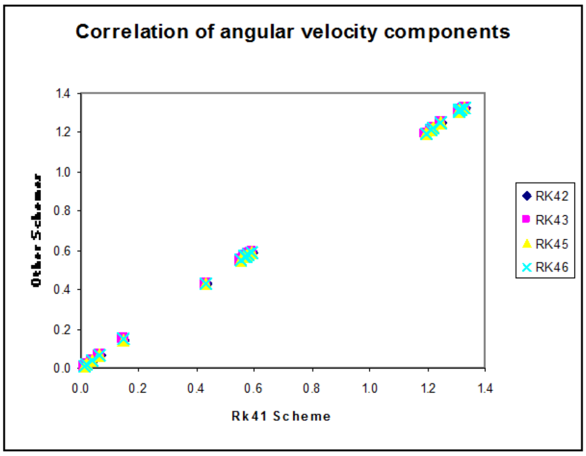


Figure 2: Comparison of the displacement component of exact and RK44 solutions of linear oscillator dynamics.

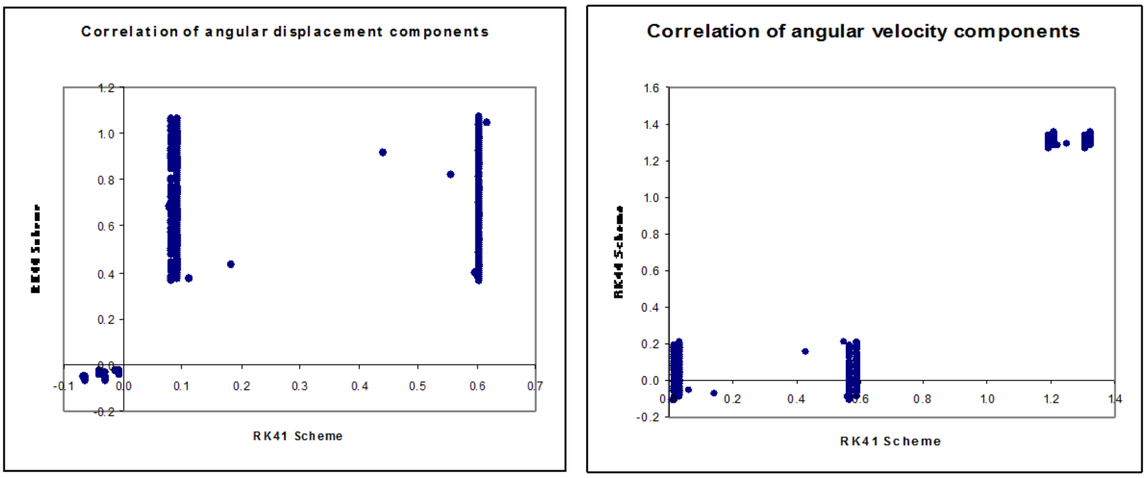
Figure 3 refers; there is good correlation of the simulated angular displacement and velocity components of the nonlinear pendulum for all the schemes with the exception of RK44 scheme. The reason for this is the scheme instability as noted for the case of linear oscillator and validated by [3] reports. Therefore, any periodically behaving nonlinear dynamics can be taken as equivalent to linear dynamic system.



(a)



(b)



(c) **(d)**

Figure 3: Correlation of steady simulated nonlinear pendulum angular displacement and velocity components for parameters $(q, g \equiv 2, 1.47)$ combination.

Table 7: First twenty steady consecutive simulated Poincare results of nonlinear pendulum for both displacement and velocity components at $(q, g \equiv 4, 1.5)$ and $(h = T_D/500)$

No of T_D	Simulated Results by Fourth Order Schemes											
	RK41		RK42		RK43		RK44		RK45		RK46	
	θ_1	θ_2	θ_1	θ_2	θ_1	θ_2	θ_1	θ_2	θ_1	θ_2	θ_1	θ_2
1	2.19	-0.05	2.20	-0.04	-1.32	1.41	1.72	-0.30	3.04	0.62	-1.32	1.41
2	1.60	2.48	1.96	2.06	-1.85	2.02	0.03	-0.40	-0.03	-0.19	-1.80	2.06
3	1.37	2.20	1.62	1.89	-0.44	0.49	1.73	-0.33	1.55	-0.43	-0.48	0.56
4	-2.61	1.45	-2.46	1.35	-2.35	1.17	0.08	-0.42	1.18	-0.52	-1.70	1.46
5	-0.92	2.57	-0.37	2.74	0.78	2.65	1.80	-0.28	2.75	0.39	2.92	0.75
6	-0.46	0.32	2.29	0.12	2.08	1.88	-0.13	-0.31	-1.23	1.54	0.55	-0.45
7	2.89	0.40	-2.08	1.85	1.77	1.74	1.54	-0.43	2.11	0.14	2.54	0.22
8	-1.23	1.43	-0.05	0.03	2.42	0.19	1.00	-0.56	1.51	2.26	-0.81	1.26
9	-2.68	1.34	1.27	-0.56	-0.49	0.71	2.80	0.43	1.80	1.70	0.81	2.65
10	0.06	2.77	2.81	0.43	-0.19	-0.11	-1.29	1.42	2.87	0.54	1.95	1.73
11	-0.48	1.52	-1.22	1.42	1.51	-0.47	2.74	0.60	-1.01	1.11	0.41	-0.06
12	-3.01	1.05	-2.59	1.41	1.57	-0.39	-1.12	1.21	1.32	2.36	1.15	-0.57
13	1.62	1.92	-1.02	2.51	0.91	-0.58	0.31	2.80	1.88	1.71	3.03	0.61
14	-0.66	1.94	-0.53	0.38	-3.09	0.74	2.02	2.15	-0.86	2.08	0.29	-0.39
15	-0.53	0.65	-2.51	1.05	0.21	0.08	1.26	2.33	-0.67	0.84	2.07	-0.14
16	-0.76	0.60	1.21	2.39	1.01	-0.61	2.06	1.99	0.87	2.78	2.94	0.41
17	0.37	-0.43	2.00	1.79	-2.95	0.86	1.54	2.00	1.57	1.84	-1.13	1.29
18	2.27	0.00	2.03	1.97	1.87	1.78	-0.48	1.33	1.37	-0.39	-0.17	2.77
19	1.25	2.46	1.60	1.87	-0.02	0.32	0.22	2.81	1.54	-0.41	-1.04	2.10
20	1.74	1.70	-2.58	1.24	1.43	-0.53	1.82	2.37	1.10	-0.55	-0.68	0.83

Tables 7 and 8 refer. The simulated results lack correlation among arbitrarily selected paired schemes for both displacement and velocity components. This is a good indication of chaotic response of the nonlinear pendulum for

$(q, g \equiv 4{,}1.5)$ combination which is supported by [23]. It is however interesting to note that all the six schemes produced qualitatively the same Poincare section when compared with figure 4 obtained for RK41 which equally compared qualitatively well with the report of [2]. Thus the Poincare solution is insensitive to choice of simulation scheme and simulation time step. But why do step by step results differ and the overall results of stable and unstable schemes indistinguishable (qualitatively) for chaotic dynamics?

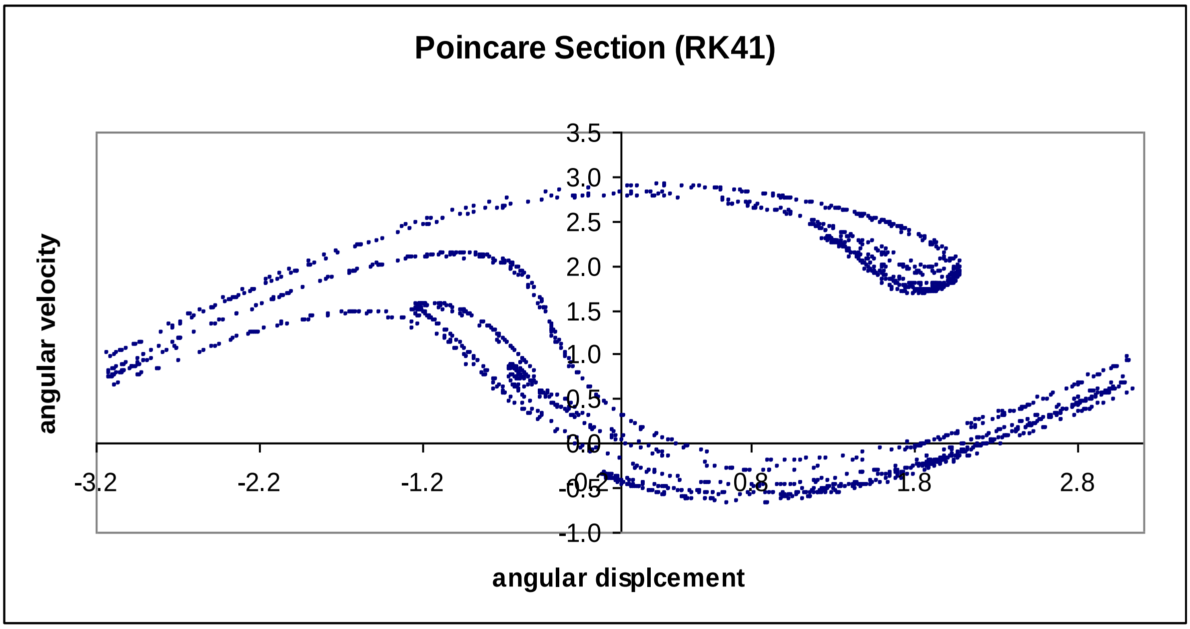


Figure 4: Poincare plot at steady state of the nonlinear pendulum for parameters $(q, g \equiv 4{,}1.5)$ and $(h = T_D/500)$ combination.

Table 8: First twenty steady consecutive simulated Poincare results of nonlinear pendulum for both displacement and velocity components at $(q, g \equiv 4{,}1.5)$ and $(h = T_D/1000)$

No of T_D	Simulated Results by Fourth Order Schemes											
	RK41		RK42		RK43		RK44		RK45		RK46	
	θ_1	θ_2	θ_1	θ_2	θ_1	θ_2	θ_1	θ_2	θ_1	θ_2	θ_1	θ_2
1	2.22	-0.05	2.22	-0.04	-0.69	0.49	2.31	0.04	2.02	-0.18	-0.69	0.49
2	1.84	1.98	1.82	1.98	-1.26	1.31	-0.28	2.78	0.67	-0.68	-1.26	1.31
3	1.90	1.73	1.93	1.75	-0.64	2.67	-1.01	2.10	-2.86	0.93	-0.71	2.65
4	-0.46	1.39	0.69	-0.17	0.58	-0.46	-0.72	0.86	1.68	1.86	0.23	-0.34
5	-0.11	2.78	1.39	-0.48	2.51	0.18	1.95	2.13	-0.88	2.08	1.89	-0.26
6	-0.69	1.99	2.01	-0.15	-0.72	1.13	1.44	2.11	-0.68	0.84	-0.08	-0.39
7	-0.60	0.76	1.03	-0.66	1.48	2.22	0.60	-0.14	1.08	2.71	1.91	-0.25
8	-1.38	2.01	-2.74	1.04	1.97	1.76	1.20	-0.54	1.50	1.98	-0.07	-0.39
9	-0.53	0.63	1.50	2.13	0.51	2.87	2.67	0.33	-1.52	1.97	1.93	-0.23
10	-0.99	1.01	1.08	2.73	1.78	1.67	-1.21	1.55	-0.46	0.53	-0.04	-0.42
11	1.64	2.05	1.50	1.98	0.66	-0.47	1.50	-0.18	-1.89	1.40	2.00	-0.19
12	1.98	2.17	-1.49	1.98	2.66	0.30	0.23	-0.50	-2.57	1.43	0.45	-0.64
13	1.37	2.20	-0.48	0.55	-1.12	1.54	2.25	-0.01	-1.14	2.46	3.09	0.65
14	-0.81	2.69	-1.63	1.45	2.01	0.07	1.54	2.27	-0.63	0.58	2.32	0.13
15	0.55	-0.44	2.80	0.65	-0.70	1.98	1.77	1.73	-1.04	1.08	1.95	0.05
16	2.47	0.16	-0.92	0.98	-0.55	0.71	2.76	0.47	1.44	2.28	-1.15	1.22
17	-0.50	0.75	1.73	1.96	0.16	-0.33	-1.25	1.45	1.95	1.74	0.71	2.69

18	0.87	-0.49	1.35	2.62	1.87	-0.26	2.96	0.79	0.61	-0.14	2.04	1.82
19	2.81	0.43	1.39	2.15	-0.08	-0.37	-2.11	1.61	1.33	-0.50	1.96	1.77
20	-1.22	1.46	-0.10	0.46	1.90	-0.25	0.96	-0.31	2.30	0.05	0.38	2.88

4. CONCLUSIONS

This study has shown that good correlation always exist between arbitrarily selected paired fourth order Runge-Kutta schemes simulated results of linear and periodically behaving nonlinear dynamics provided the schemes stability is established. That simulated results gross inaccuracy associated with scheme instability can be drastically reduced with relatively smaller computation time step, but at greater computation efforts. Furthermore, chaotically responding dynamics can be differentiated by its simulated results lacking in correlation between arbitrarily selected pair schemes and having Poincare results that are qualitatively the same. Thus the study results can be adopted as dynamics systems characterizing tool.

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