International Journal of Engineering and Technology Volume 4 No. 11, November, 2014

On Matching Characterization of Graphs

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ABSTRACT

We consider of the problem of graphs characterization by its matching polynomial. In the paper, we show that $P_m \cup T(1,3,n)$ are determined by its matching polynomial iff m is even or 3 and $n \neq 3,6,11$.

Keywords：T *-shape tree*；*Matching polynomial*；*Matching Characterization*

1. INTRODUCTION

All graphs in the paper are finite and have no loops or multiple edges .Let G be a graph with n vertices.An r-matching in a graph G is a set of r edges,no two of which have a vertex in common.The number of r-matching in G will be denoted by $p(G,r)$ .We set $p(G,0)=1$ and define the matching polynomial of G by

$$\mu(G,x)=\sum_{r\geq 0}(-1)^r p(G,r)x^{n-2r}$$

For any graph G ,the roots of $\mu(G,x)$ are all real numbers.Assume that $\gamma_1(G)\geq\gamma_2(G)\geq\cdots\geq\gamma_n(G)$,the largest root $\gamma_1(G)$ is referred to as the largest mathing root of G.

Throughout the paper ,we denote by P_n and C_n the path and the cycle on n vertices, respectively. $T(a,b,c)(a\leq b\leq c)$ denotes the tree with a vertex v of degree 3 such that $T(a,b,c)-v=P_a\cup P_b\cup P_c$,and $H(a,b,c)$ denotes the tree obtained form the path with versices $1,2,\ldots,a+b+c-1$ (in order) by attching a pendant edge at each of the verices a and $a+b$. $Q(s_1,s_2)$ is obtained by appending a cycle C_{s_1+1} to a pendant vertex of a path P_{s_2} .Two graphs are matching equivalency if they share the same matching polynomial.A graph G is said to be determined by its matching polynomial if for any graph H, $\mu(G,x)=\mu(H,x)$ implies that H is isomorphic to G.

The study in this ares has made great progress. For details, the reader is referred to the surveys[4-7]. In the paper ,we prove $P_m\cup T(1,3,n)$ are determined by its matching polynomial iff m is even or 3 and $n\neq 3,6,11$.

2. BASIC RESULTS

Lemma 2.1[1]. The matching polynomial $\mu(G,x)$ satisfies the following identities:

(a) $\mu(G\cup H,x)=\mu(G,x)\mu(H,x)$.

(b) $\mu(G,x)=\mu(G\setminus e,x)-\mu(G\setminus u,v,x)$ if $e=\{u,v\}$ is an edge of G.

Lemma 2.2[1].Let G be a connected graph,and let H be a proper subgraph G. Then $\gamma_1(G)>\gamma_1(H)$.

Lemma 2.3[1].Let u be a vertex in the graph G .Then the roots of $\mu(G\setminus u,x)$ interlace those of $\mu(G,x)$. If G is connected then the largest matching root of $\mu(G,x)$ is simple ,and is strictly greater than the largest matching root of $\mu(G\setminus u,x)$.

Lemma 2.4[2,3].Let G be a connected graph, then

(a) $\gamma_1(G)<2$ iff $G\in\{P_n,T(1,1,n),T(1,2,2),T(1,2,3),T(1,2,4),C_m,Q(2,1)\}$

.

(b) $\gamma_1(G)=2$ iff $G\in\{K_{1,4},T(2,2,2),T(1,3,3),T(1,2,5),I_m,Q(2,2),Q(3,1)\}$

.

Lemma 2.5 [2,3]. The connected graphs G with the largest matching root in the interval $(2,\sqrt{2+\sqrt{5}})$ iff it are precisely the graphs of the following types:

(a) $T(1,2,c)(c\geq 6)$, $T(1,b,c)(c>b>2)$, $T(2,2,c)(c>2)$, $T(2,3,3)$.

(b) $Q(2,n)(n\geq 3)$, $Q(m,1)(m\geq 4)$, $Q(3,2)$.

(c) $H(a,b,c)$,for $(a,b,c,)\in\{(2,1,3),(3,4,3),(3,5,4),(4,7,4),(4,8,5)\}$ or $a>1$

$b\geq b^*(a,c), c>1$ where $(a,c)\neq(2,2)$ and

$$b^*(a,c)=\begin{cases} a+c, a>3,\\ 2+c, a=3,\\ -1+c, a=2,\end{cases}$$

Lemma 2.6 [3].Let G be a tree and let $G_{u,v}$ be obtained from G by subdividing the edge uv of G, then

(a) $\gamma_1(G_{u,v})>\gamma_1(G)$ if uv not lies on an internal path of G.

(b) $\gamma_1(G_{u,v})<\gamma_1(G)$ if uv lies on an internal path of G ,and if G is not isomorphic to $H(2,m,2)$.

Lemma 2.7 [4] . $\gamma_1(T(1,2,n))<\gamma_1(T(1,3,5))$, $\gamma_1(T(1,3,n))<\gamma_1(T(1,4,6))$.

Lemma 2.8. $\gamma_1(T(1,m,n))<\gamma_1(H(a,b,c))(a\geq 2, c\geq m+1)$.

Proof. If $b\geq n-1$, clearly $T(1,m,n)$ is a proper subgraph of $H(2,b,m+1)$, By Lemma 2.2, $\gamma_1(T(1,m,n))<\gamma_1(H(2,b,m+1))$. If $b<n-1$, $T(1,m,n)$ is a proper subgraph of $H(2,n-1,m+1)$,then $\gamma_1(T(1,m,n))<\gamma_1(H(2,n-1,m+1))$.

By Lemma2.6, $\gamma_1(H(2,n-1,m+1))<\gamma_1(H(2,b,m+1))$. So, $\gamma_1(T(1,m,n))<\gamma_1(H(2,b,m+1))$ and $H(2,b,m+1)$ is a proper subgraph of $H(a,b,c)$ $(a\geq 2, c\geq m+1)$, By Lemma 2.2,thus the lemma holds.

Lemma 2.9.

If $\gamma_1(T(1,3,n))=\gamma_1(H(2,m,3))=\gamma_1(H(3,b,3))$,then $b=2m+2$ and n may only be 5,6,7,11.

Proof.

By Lemma 2.1, $x\mu(H(3,2m+2,3),x)=\mu(H(2,m,3),x)\cdot\mu(T(1,2,m),x)$

So,we have $\gamma_1(H(2,m,3))=\gamma_1(H(3,2m+2,3))$ .Direct calculate the largest matching root of $T(1,3,n)(n=4,5,6,7,11)$ and $H(2,b,3)(b=2,3,4,5,8)$ (using Matlab 8.0),we immediately have the following:

$$\gamma_1(T(1,3,5))=\gamma_1(H(2,8,3))=\gamma_1(H(3,18,3))$$
$$\gamma_1(T(1,3,6))=\gamma_1(H(2,5,3))=\gamma_1(H(3,12,3))$$
$$\gamma_1(T(1,3,7))=\gamma_1(H(2,4,3))=\gamma_1(H(3,10,3))$$
$$\gamma_1(T(1,3,11))=\gamma_1(H(2,3,3))=\gamma_1(H(3,8,3))$$

By Lemma 2.8, If $n\leq 4$, $\gamma_1(T(1,3,3)<\gamma_1(T(1,3,4))=\gamma_1(T(1,2,9))<\gamma_1(H(2,b,3))$

.

If $5\leq n\leq 11$, $\gamma_1(H(2,4,3)\leq\gamma_1(T(1,3,n))\leq\gamma_1(H(2,3,3))$. If $n\geq 12$,

$\gamma_1(H(2,3,3))=\gamma_1(T(1,3,11))<\gamma_1(T(1,3,n))<\gamma_1(T(1,4,6))<\gamma_1(H(2,2,3)=2.0421$

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This completes the proof .

Lemma2.10 [5] . Let $G=T(a,b,c)$, then $\gamma_1(Q(a+1,b-1)) \le \gamma_1(T(a,b,c)) \le \gamma_1(Q(a+1,b)) < \gamma_1(Q(s,t))(s>a,t>b)$,iff $c=b, a+b+2$ with equality.

Lemma 2.11 [6]**.** Let G be a graph with largest mathing root $\gamma_1(G)<2$,then G be determined by its matching polynomial iff

$$G = kK_1 \cup m_2P_2 \cup m_3P_3 \cup (\bigcup_{i\ge 2} m_{2i}P_{2i}) \cup (\bigcup_{j\ge 3} n_jC_j) \cup dD_4 \cup eT(1,2,3) \cup fT(1,2,4)$$

.

Where

$$kn_j = m_jn_{i+1} = m_2d = m_3d = n_3e = n_{15}e = n_3n_5f = n_5n_9f = 0$$

and k, m_i, n_j,

$d,e,f \in N$.

3. MAIN RESULTS

Theorem3.1. Let $G=P_m \cup T(1,3,n)$ .Then G is uniquely determined by its matching polynomial iff m is even or 3 and $n \ne 3,6,11$.

Proof. The necessary condition follows immediately from Lemma 2.1.

We have

$$\mu(P_{2k+1} \cup T(1,3,n),x)(k\ge 2) = \mu(P_k,x)\mu(C_{k+1},x)\mu(T(1,3,n),x) = \mu(P_k \cup C_{k+1} \cup T(1,3,n),x)$$

$$\mu(P_m \cup T(1,3,3),x) = \mu(P_m,x)\mu(P_3,x)\mu(Q(3,1),x) = \mu(P_m \cup P_3 \cup Q(3,1),x)$$

$$\mu(P_m \cup T(1,3,6),x) = \mu(P_m,x)\mu(P_5,x)\mu(Q(2,3),x) = \mu(P_m \cup P_5 \cup Q(2,3),x)$$

$$\mu(P_m \cup T(1,3,11),x) = \mu(P_m,x)\mu(C_5,x)\mu(T(1,4,5),x) = \mu(P_m \cup C_5 \cup T(1,4,5),x)$$

Now suppose that m is even or 3 and $n \ne 3,6,11$, H is a graph being matching equivalency with G ,hance $\gamma_1(H)=\gamma_1(G)=\gamma_1(T(1,3,n))$.We proceed to prove that H must be isomorphic to G .,By Lemma 2.5,2.7, $\gamma_1(H) \in (2,\ \gamma_1(T(1,4,6))) \subset (2,\ \sqrt{2+\sqrt{5}})$. Set

$\varpi_1 = \{T(1,2,c)(c\ge 6), T(2,2,c)(c\ge 3), T(1,b,c)(b\ge 3), T(2,3,3)\}$,

$\varpi_2 = \{H(2,4,3), H(2,8,3), H(3,10,3), H(3,18,3)\}$,

$\varpi_3 = \{T(1,1,c), T(1,2,2), T(1,2,3), T(1,2,4)\}$, then by Lemma2.8,2.9,2.10 ,we get

$$H = t_1 * \varpi_1 \cup t_2 * \varpi_2 \cup t_3 * \varpi_3 \cup t_4 \cdot K_1 \cup t_5 \cdot Q(2,1) \cup (\bigcup_{i=0}^{s} P_{n_i}) \cup (\bigcup_{i=0}^{l} C_{p_i})$$

(where $t_i * \varpi_i\ (i=1,2,3)$ denotes t_i re-elements form ϖ_i , $n_i \ge 2, p_i \ge 3$).By Lemma2.3, $t_1+t_2=1$. We distinguish the following cases:

Case 1. If $t_1=1$. We further cndsider several cases:

Subcase 1. $H = T(1,2,c)(c\ge 6) \cup t_3 * \varpi_3 \cup t_4 \cdot K_1 \cup t_5 \cdot Q(2,1) \cup (\bigcup_{i=0}^{s} P_{n_i}) \cup (\bigcup_{i=0}^{l} C_{p_i})$

Then

$\gamma_1(G) = \gamma_1(T(1,3,n)) = \gamma_1(H) = \gamma_1(T(1,2,c))(c\ge 6)$.

By Lemma2.7, $n=4$.

Direct computation shows $\mu(T(1,2,9),x) = \mu(T(1,3,4) \cup C_4, x)$,hence we have P_m matching equivalency with $C_4 \cup t_3 * \varpi_3 \cup t_4 \cdot K_1 \cup t_5 \cdot Q(2,1) \cup (\bigcup_{i=0}^{s} P_{n_i}) \cup (\bigcup_{i=0}^{l} C_{p_i})$

Which is a contradiction.

Subcase 2. $H = T(2,2,c)(c\ge 3) \cup t_3 * \varpi_3 \cup t_4 \cdot K_1 \cup t_5 \cdot Q(2,1) \cup (\bigcup_{i=0}^{s} P_{n_i}) \cup (\bigcup_{i=0}^{l} C_{p_i})$

Then

$\gamma_1(G) = \gamma_1(T(1,3,n)) = \gamma_1(H) = \gamma_1(T(2,2,c))(c\ge 3)$.

By Lemma 2.1 ,we get

$\mu(T(2,2,c),x) = \mu(Q(2,c) \cup P_3, x) = \mu(c+1,1),x)$,which is a contradiction.

Subcase 3. $H = T(2,3,3)$

$\bigcup t_3 * \varpi_3 \bigcup t_4 \cdot K_1 \bigcup t_5 \cdot Q(2,1) \bigcup (\bigcup_{i=0}^{s} P_{n_i}) \bigcup (\bigcup_{i=0}^{l} C_{p_i})$ .T

he

same argument as subcase 2 can be used to get a contradiction.

Subcase 4. $H = T(1,b,c)(b \geq 3)$

$\bigcup t_3 * \varpi_3 \bigcup t_4 \cdot K_1 \bigcup t_5 \cdot Q(2,1) \bigcup (\bigcup_{i=0}^{s} P_{n_i}) \bigcup (\bigcup_{i=0}^{l} C_{p_i})$.

Then

$\gamma_1(G) = \gamma_1(T(1,3,n)) = \gamma_1(H) = \gamma_1(T(1,b,c))(b \geq 3)$.

By Lemma 2.7 ,2.11 and

$n \neq 11$,we have

$b = 3, c = n, t_3 = t_4 = t_5 = p_i = 0, s = 0, n_0 = m$,thus

H be

isomorphic to G.

Case 2. If $t_2 = 1$. By lemma 2.9,we get $n = 5$ or 7. First suppose that $n =$ 5. Note that

$\mu(K_1 \bigcup H(3,18,3), x) = \mu(T(1,3,5) \bigcup Q(2,1) \bigcup T(1,2,8), x)$

and

$\mu(H(2,8,3), x) = \mu(T(1,3,5) \bigcup Q(2,1), x)$ .Thus we have

$\mu(K_1 \bigcup P_m, x) =$

$\mu(T(1,2,8) \bigcup t_3 * \varpi_3 \bigcup t_4 \cdot K_1 \bigcup t_5 \cdot Q(2,1) \bigcup (\bigcup_{i=0}^{s} P_{n_i}) \bigcup (\bigcup_{i=0}^{l} C_{p_i}), x)(t_5 \geq 1)$

or $\mu(P_m, x) =$

$\mu(t_3 * \varpi_3 \bigcup t_4 \cdot K_1 \bigcup t_5 \cdot Q(2,1) \bigcup (\bigcup_{i=0}^{s} P_{n_i}) \bigcup (\bigcup_{i=0}^{l} C_{p_i}), x)(t_5 \geq 1)$

. Which contdadicts to Lemma 2.11.Secondly assume $n = 7$. By Lemma 2.1,direct Direct computation shows $\mu(P_3 \bigcup H(3,10,3), x) = \mu(T(1,3,7) \bigcup T(1,2,4), x)$ and

$\mu(P_3 \bigcup H(2,4,3), x) =$

$\mu(K_1 \bigcup T(1,3,7), x)$,hance we get

$\mu(P_3 \bigcup P_m, x) =$

$\mu(T(1,2,4) \bigcup t_3 * \varpi_3 \bigcup t_4 \cdot K_1 \bigcup t_5 \cdot Q(2,1) \bigcup (\bigcup_{i=0}^{s} P_{n_i}) \bigcup (\bigcup_{i=0}^{l} C$

or

$\mu(P_3 \bigcup P_m, x) =$

$\mu(t_3 * \varpi_3 \bigcup t_4 \cdot K_1 \bigcup t_5 \cdot Q(2,1) \bigcup (\bigcup_{i=0}^{s} P_{n_i}) \bigcup (\bigcup_{i=0}^{l} C_{p_i}), x)(t_4 \geq 1)$

. which is a contradiction.

Combing cases1,2 , H is isomorphic to G .The proof is complete.

For a graph ,its matching polynomial determine the matching polynomial of its complement[7], so the complement of $G = P_m \bigcup T(1,3,n)$ is determined by its matching polynomial iff m is even or 3 and $n \neq 3,6,11$.

REFERENCES

[1]. Godsil C.D. Algebraic Combinatorics[M] , New York ,London:Chapman and Hall 1993.

[2]. N.Ghareghani,G.R.Omidi, B.Tayfeh-Rezaie,Spectral Characterization of graphs with index at most $\sqrt{2+\sqrt{5}}$ [J] , Linear Algebra Appl.420(2007):483-489

[3]. D.M.Cvetkovic, M.Doob,H.Sachs, Spectra of Graphs[M], Academic Press, New York,1980. Third edition Johann Ambrosius ,Barth,Verlag 1995.

[4]. Shichang Shen, A note on matching uniqueness of $T(1,3,n)$ and its complement[J] ,Journal of Qinghai Normal University, 2006(2):4-6.

[5]. Shichang Shen,The matching uniqueness T-shape tree with nearly equal length[J].Pure and Applied Mathematics , ,2007,2(3):297-301.

[6]. Haicheng Ma. The matching equivalent classes of graphs with the maximum root less than 2[J]. Journal of Systems Science and Mathematical Sciences 2003,(3):337-342.

[7]. Beezet R.A.,Farrell E.J., The matching polynomials of a regular graphs[J], discrete. Math., 1995,137:7-18.